Borel hierarchies in infinite products of Polish spaces

RANA BARUA and ASHOK MAITRA*

Stat-Math Division, Indian Statistical Institute, Kolkata 700 108, India
*School of Statistics, University of Minnesota, Minneapolis, MN, USA
E-mail: rana@isical.ac.in; maitra@stat.umn.edu

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Abstract. Let $H$ be a product of countably infinite number of copies of an uncountable Polish space $X$. Let $\Sigma_\xi$ ($/\Sigma_\xi$) be the class of Borel sets of additive class $\xi$ for the product of copies of the discrete topology on $X$ (the Polish topology on $X$), and let $\mathcal{B} = \bigcup_{\xi < \omega_1} \Sigma_\xi$.

We prove in the Lévy–Solovay model that $\Sigma_\xi = /\Sigma_\xi \cap \mathcal{B}$ for $1 \leq \xi < \omega_1$.

Keywords. Borel sets of additive classes; Baire property; Levy–Solovay model; Gandy–Harrington topology.

1. Introduction

Suppose $X$ is a Polish space and $N$ the set of positive integers. We consider $H = X^N$ with two product topologies: (i) the product of copies of the Polish topology on $X$, so that $H$ is again a Polish space and (ii) the product of copies of the discrete topology on $X$. Define now the Borel hierarchy in the larger topology on $H$. To do so, we need some notation. An element of $H$ will be denoted by $h = (x_1, x_2, \ldots, x_n, \ldots)$ and for $m \in N$, $p_m(h)$ will denote the first $m$ coordinates, that is, $p_m(h) = (x_1, x_2, \ldots, x_m)$. For $n \in N$ and $A \subseteq X^n$, $\text{cyl}(A)$ will denote the cylinder set with base $A$, that is,

$$\text{cyl}(A) = \{h \in H : p_n(h) \in A\}.$$

The Borel hierarchy for the larger topology on $H$ can now be defined as follows:

$$\Sigma_0 = \Pi_0 = \{\text{cyl}(A) : A \subseteq X^n, \ n \geq 1\}$$

and for $\xi > 0$,

$$\Sigma_\xi = \left(\bigcup_{\eta < \xi} \Pi_\eta\right)_{\sigma}, \quad \Pi_\xi = \neg \Sigma_\xi.$$

The Borel hierarchy on $H$ with respect to the smaller topology is defined in the usual way:

$$\Sigma_1 = \{V : V \text{ is open in } H \text{ in the smaller topology}\}, \quad \Pi_1 = \neg \Sigma_1.$$
and, for $\xi > 1$,

$$\Sigma_{\xi} = \left( \bigcup_{\eta < \xi} \Pi_{\eta} \right)_{\sigma}; \quad \Pi_{\xi} = \neg \Sigma_{\xi}.$$ 

Let

$$B = \bigcup_{\xi < \omega_1} \Sigma_{\xi} = \bigcup_{\xi < \omega_1} \Pi_{\xi}.$$ 

The problem we will address in this article is whether

$$\Sigma_{\xi} = \Sigma_{\xi} \cap B \quad \text{for} \quad 1 \leq \xi < \omega_1. \quad (*)$$

To tackle the problem we will use the methods of effective descriptive set theory. We therefore have to formulate the lightface version of $(*)$. We refer the reader to [Mo] and [L1] for definitions of lightface concepts. We take $X$ to be the recursively presentable Polish space $\omega^\omega$ hereafter.

Define

$$\Sigma_0^* = \Pi_0^* = \{\text{cyl}(A): A \text{ is } \Delta_1^1 \text{ in } (\omega^\omega)^n, n \geq 1\},$$

and, for $1 \leq \xi < \omega_1^{ck}$,

$$\Sigma_\xi^* = \bigcup_1^1 \left( \bigcup_{\eta < \xi} \Pi_\eta^* \right)$$

and

$$\Pi_\xi^* = \neg \Sigma_\xi^*,$$

where $\bigcup_1^1 \left( \bigcup_{\eta < \xi} \Pi_\eta^* \right)$ is a $\Delta_1^1$ union of members of $\bigcup_{\eta < \xi} \Pi_\eta^*$. The lightface analogue of $(*)$ is then

$$\Sigma_\xi^* = \Delta_1^1 \cap \Sigma_\xi, \quad \text{for} \quad 1 \leq \xi < \omega_1^{ck}. \quad (***)$$

In order to state the main result of the article, we equip $\omega^\omega$ with the Gandy–Harrington topology, that is, the topology whose base is the pointclass of $\Sigma_1^1$ sets. The key property of this topology is that it satisfies the Baire category theorem (see [L1]). Consider now the following statement of set theory:

**O** Every subset of $\omega^\omega$ has the Baire property with respect to the Gandy–Harrington topology. The main result of the article can now be stated.

**Theorem 1.1.** Assume **O**. Let $1 \leq \xi < \omega_1^{ck}$. If $A$ and $B$ are $\Sigma_1^1$ subsets of $H$ such that $A$ can be separated from $B$ by a $\Sigma_\xi$ set, then $A$ can be separated from $B$ by a $\Sigma_\xi^*$ set.

An immediate consequence is

**Corollary 1.2**

**O** implies $(***)$. 