**AT-algebras and extensions of AT-algebras**

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**Abstract.** Lin and Su classified $\text{AT}$-algebras of real rank zero. This class includes all $\text{AT}$-algebras of real rank zero as well as many $\text{C}^*$-algebras which are not stably finite. An $\text{AT}$-algebra often becomes an extension of an $\text{AT}$-algebra by an $\text{AF}$-algebra. In this paper, we show that there is an essential extension of an $\text{AT}$-algebra by an $\text{AF}$-algebra which is not an $\text{AT}$-algebra. We describe a characterization of an extension $E$ of an $\text{AT}$-algebra by an $\text{AF}$-algebra if $E$ is an $\text{AT}$-algebra.

**Keywords.** $\text{AF}$-algebra; $\text{AT}$-algebra; $\text{AT}$-algebra; extension; index map.

1. Introduction

During the development of classification of nuclear separable $\text{C}^*$-algebras, a special class of inductive limits of finite direct sums of matrix algebras over $T$-algebras was classified by Lin and Su [6], where $T$-algebras are unital essential extensions of $\text{C}(S^1)$ by compact operators $K$:

$$0 \longrightarrow K \longrightarrow E \longrightarrow \text{C}(S^1) \longrightarrow 0.$$  

Each $\text{C}^*$-algebra in this special class is said to be an $\text{AT}$-algebra. One of the important features which makes $\text{AT}$-algebras essentially different from $\text{AH}$-algebras is that the torsion in $K_0$ does not arise from the torsion parts of certain metric spaces but from nontrivial extensions of $\text{C}(S^1)$ by $K$. Let $A$ be an $\text{AT}$-algebra. The invariant consists of the abelian semigroup $V(A)$, the Murry–von Neumann equivalence classes of projections in matrices of $A$, an abelian semigroup $k(A)^+$, some equivalence classes of homotopy classes of hyponormal partial isometries in matrices of $A$ and a homomorphism $d$ from $k(A)^+$ into $V(A)$. The main result of [6] states that the above invariant, together with the class of the identity, is complete for the class of $\text{C}^*$-algebras.

An $\text{AT}$-algebra often becomes an essential extension of an $\text{AT}$-algebra by an $\text{AF}$-algebra. Consequently, a question of whether an essential extension of an $\text{AT}$-algebra by an $\text{AF}$-algebra is an $\text{AT}$-algebra, is raised. In this paper, we show that there is an essential extension of an $\text{AT}$-algebra by an $\text{AF}$-algebra which is not in the class. Recently there have been rapid advances in the study of quasidiagonal extensions of $\text{C}^*$-algebras (c.f. [2]). Tracially quasidiagonal extensions are studied by Lin in [4]. In [1], Brown and Dadarlat show that the index maps $\delta_0$ and $\delta_1$ of a quasidiagonal extension of $\text{C}^*$-algebras are zero.
In [5], Lin and Rørdam show that if $E$ is an extension of an $\mathcal{AT}$-algebra by an $\mathcal{AT}$-algebra and $E$ has real rank zero, then $E$ is an $\mathcal{AT}$-algebra if and only if the index maps are both zero. Accordingly, in this paper, we attempt to describe a characterization of an extension $E$ of an $\mathcal{AT}$-algebra by an AF-algebra if $E$ is an $\mathcal{AT}$-algebra via the index maps.

2. $\mathcal{AT}$-algebra as an essential extension of an $\mathcal{AT}$-algebra

Let $C(S^1)$ be the continuous functions on the unit circle and let $\mathcal{K}$ be the compact operators on an infinite dimensional separable Hilbert space. $\mathcal{T}_k$ is an essential unital extension of $C(S^1)$ by $\mathcal{K}$ with index $−k ∈ \mathbb{Z}$:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T}_k \longrightarrow C(S^1) \longrightarrow 0.$$  

It is well-known that two extensions with the same index are isomorphic as $C^*$-algebras. We call these algebras $T$-algebras. It is obvious that $\mathcal{T}_k$ is isomorphic to $\mathcal{T}_{−k}$. We consider only those $\mathcal{T}_k$ with $k ≥ 0$. We now give another description of $\mathcal{T}_k$ (for $k ≥ 0$). Let $S_0$ be an unitary in $\mathcal{B}(H)$ with essential spectrum $S^1$. Then $\mathcal{J}_0$ is isomorphic to the $C^*$-subalgebra of $\mathcal{B}(H)$ generated by $S_0$ and $K(H)$. Let $S_1$ be the standard unilateral shift operator acting on the Hilbert space $H = l^2$. Then $\mathcal{T}_k (k > 0)$ is isomorphic to the $C^*$-subalgebra of $\mathcal{B}(H)$ generated by $(S_1)^k$ and $K(H)$.

Lemma 2.1. Let $A$ be a $C^*$-algebra with an approximate unit of projections, $\{I_\lambda\}_{\lambda \in \Lambda}$ a set of ideals of $A$. If quotient $A/I_\lambda$ is a finite $C^*$-algebra for each $\lambda \in \Lambda$, then $A/ \bigcap_{\lambda \in \Lambda} I_\lambda$ is a finite $C^*$-algebra.

Proof. Let $\{p_i\}$ be an approximate unit of projections in $A$ and $\pi$ be the quotient map from $A$ to $A/ \bigcap_{\lambda \in \Lambda} I_\lambda$. Then $\pi (p_i)$ becomes an approximate unit of projections in $A/ \bigcap_{\lambda \in \Lambda} I_\lambda$. For any $i$, we assume that $v^*v = \pi (p_i)$. There is $w \in p_iAp_i$ such that $\pi (w) = v$. Since $\pi (w^*w) = \pi (p_i)$, $v^*w = p_i + \bigcap_{\lambda \in \Lambda} I_\lambda$. By the hypothesis of the lemma, $uw^* \in p_i + \bigcap_{\lambda \in \Lambda} I_\lambda$, so $ww^* = \pi (w^*w) = \pi (p_i)$, Therefore $A/ \bigcap_{\lambda \in \Lambda} I_\lambda$ is a finite $C^*$-algebra.

DEFINITION 2.2

Let $A$ be a $C^*$-algebra with an approximate unit of projections. By the above lemma, there exists the smallest ideal $I$ of $A$ such that $A/I$ is a finite $C^*$-algebra. We denote this ideal by $I(A)$, and denote $A/I$ by $Q(A)$.

Lemma 2.3. Let $E = \lim_{\rightarrow} (E_n, \phi_n)$ be an inductive limit $C^*$-algebra, where each $E_n$ is a finite direct sum of matrix algebras over $T$-algebras, and each connecting map from $E_n$ to $E_{n+1}$ satisfies the following: if $M_\lambda (\mathcal{J}_0)$ is a summand of $E_n$, then the connecting map restricted to this block vanishes on $M_\lambda (\mathcal{K})$. Then $I(E) = \lim_{\rightarrow} (I(E_n), \phi_n)$ and $Q(E) = \lim_{\rightarrow} (Q(E_n), \tilde{\phi}_n)$, where $\tilde{\phi}_n$ is the $^*$-homomorphism which is induced by $\phi_n$.

Proof. It is easy to see that $\phi_n(I(E_n)) \subset I(E_{n+1})$ for each $n$. Since $\lim_{\rightarrow} (Q(E_n), \tilde{\phi}_n)$ is a finite $C^*$-algebra, $I(E) \subset \lim_{\rightarrow} (I(E_n), \phi_n)$. It remains to show that $\phi(I(E_n)) \subset I(E)$. We may assume that $E_n = M_\lambda (\mathcal{T}_k)$ and $\phi_{n,\infty} (M_\lambda (\mathcal{K})) \neq 0$. Note that $M_\lambda (\mathcal{K})$ is the minimal ideal of $M_n (\mathcal{T}_k)$, $\phi_{n,\infty}$ is injective on $M_n (\mathcal{T}_k)$. There exists $v ∈ M_n (\mathcal{T}_k)$