Euclid’s Fifth Postulate

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In this article, we review the fifth postulate of Euclid and trace its long and glorious history. That after two thousand years, it led to so much discussion and new ideas speaks for itself.

“It is hard to add to the fame and glory of Euclid, who managed to write an all-time bestseller, a classic book read and scrutinized for the last twenty three centuries.” The book is called *The Elements* and consists of 13 books all devoted to various aspects of geometry and number theory. Of these, the most quoted is the one on the fundamentals of geometry Book I, which has 23 definitions, 5 postulates and 48 propositions. Of all the wealth of ideas in *The Elements*, the one that has claimed the greatest attention is the Fifth Postulate. For two thousand years, the Fifth postulate, also known as the ‘parallel postulate’, was suspected by mathematicians to be a theorem, which could be proved by using the first four postulates.

Starting from the commentary of Proclus, who taught at the Neoplatonic Academy in Athens in the fifth century some 700 years after Euclid to al Gauhary (9th century), to Omar Khayyam (11th century) to Saccheri (18th century), the fifth postulate was sought to be proved. Euclid himself had just stated the fifth postulate without trying to prove it. The main reason that such a proof was so much sought after, was that while other postulates appeared to be self-evident and obvious, the fifth postulate involving the intersection of lines at potentially infinite distances, was hardly self-evident. In these proofs it was unwittingly assumed that an equivalent axiom holds. This was invariably the one which was better known as Playfair’s axiom, after the Scottish mathematician John Playfair (1748 – 1819). Besides, some like al-Tusi (13th century) and Girolamo Saccheri (1667 – 1733) tried to prove the postulate by a *reductio ad absurdum* method. In 1733, Saccheri, a professor of rhetoric, theology and philosophy...
at a Jesuit college in Milan, published a two volume work entitled *Euclid free of every flaw* “Euclidus Vindicatus”. The nineteenth century saw mathematicians exploring all possible alternatives to the fifth postulate and discovering logically consistent geometries. In the 1820’s Nikolai Lobachevsky and Janos Bolyai independently realized that entirely self consistent “non-Euclidean” geometries could be created in which the parallel postulate did not hold. It is probable that Carl Friedrich Gauss had actually studied the problem before that, but if he did so, he did not publish any of his results. The names of Bernhard Riemann (1826 – 1866) and Henri Poincaré (1854 – 1912) are associated with the development of resulting geometries. By the end of the last century, it was shown once and for all that the fifth postulate is independent of the remaining postulates and that all attempts at proving it using the other four were doomed from the beginning.

1. The Postulates

The five postulates have been cited over and over again, but a repetition will do no harm. They are:

1) A straight line can be drawn joining any two points.
2) Any straight line segment can be extended indefinitely in a straight line.
3) Given any straight line segment, a circle can be drawn having the segment as radius and an endpoint as centre.
4) All right angles are congruent.
5) If two lines are drawn, which intersect a third line in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines must intersect each other on that side if extended far enough. (Figure 1.)

The fifth postulate is also known as the Parallel Postulate, because it can be used to prove properties of parallel lines. Euclid used only the first four postulates for the first 28 propositions in Book I, but was forced to invoke the parallel postulate in proving Proposition 29. This states that a straight line falling on parallel straight lines makes the alternate angles equal to one another, the