Symmetries and Conservation Laws in Classical and Quantum Mechanics

2. Quantum Mechanics

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In Part 1 of this two-part article we have spelt out, in some detail, the link between symmetries and conservation principles in the Lagrangian and Hamiltonian formulations of classical mechanics (CM). In this second part, we turn our attention to the corresponding question in quantum mechanics (QM). The generalization we embark upon will proceed in two directions: from the classical formulation to the quantum mechanical one, and from a single (infinitesimal) symmetry to a multi-dimensional Lie group of symmetries. Of course, we always have some definite physical system in mind. We also assume that the reader is familiar with the elements of quantum mechanics at the level of a standard first course on the subject. Operators will be denoted with an overhead caret, e.g., \( \hat{A}, \hat{G}, \hat{U} \), etc., while \( [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \) is the commutator of \( \hat{A} \) and \( \hat{B} \).

1. Symmetry in Quantum Mechanics

The treatment of symmetry and invariance in QM is closely modelled on the Hamiltonian formalism in CM. As is well known, we have the formal replacements

- Poisson bracket (PB) in CM \( \rightarrow \) commutator/\( (i\hbar) \) in QM.
- Canonical transformation (CT) in phase space \( \rightarrow \) unitary transformation (UT) on Hilbert space.

These statements will be qualified and elaborated upon,
subsequently. Corresponding to any continuous symmetry of a quantum system, we have a constant of the motion (COM) that is now a hermitian operator \( \hat{G} \). A finite symmetry transformation, as opposed to an infinitesimal one, is represented by a UT built up from a succession of infinitesimal transformations. It has the general form
\[
\hat{U}(\alpha) = e^{-i\alpha \hat{G}/\hbar}.
\] (1)

Here, \( \hat{G} \) is the infinitesimal generator (often abbreviated to simply ‘the generator’) of the transformation, and \( \alpha \) is the (real) parameter characterising the transformation. The constant \( \hbar \) has been introduced explicitly in the exponent in (1) for convenience – this is the form in which unitary transformations commonly occur in QM. Note that the product \( (\alpha \hat{G}) \) has the same physical dimensions as \( \hbar \), i.e., those of angular momentum, or (length) \( \times \) (linear momentum). The effects of the transformation on state vectors (or wave functions) and on dynamical variables, respectively, are given by
\[
|\Psi\rangle \rightarrow |\Psi'\rangle = \hat{U}(\alpha)|\Psi\rangle, \quad (2a)
\]
\[
\hat{A} \rightarrow \hat{A}' = \hat{U}(\alpha) \hat{A} \hat{U}^{-1}(\alpha). \quad (2b)
\]

Since \( \hat{G} \) is hermitian and \( \hat{U}(\alpha) \) is unitary, we have
\[
\hat{U}^{-1}(\alpha) = \hat{U}^\dagger(\alpha) = e^{i\alpha \hat{G}/\hbar}. \quad (3)
\]

A combination like \( \hat{U} \hat{A} \hat{U}^{-1} \) is called the conjugation of \( \hat{A} \) by \( \hat{U} \).

The equation of motion (EOM) in quantum mechanics is the Schrödinger equation for the state vector of a system, namely,
\[
i\hbar \frac{d|\Psi(t)\rangle}{dt} = \hat{H}(t)|\Psi(t)\rangle, \quad (4)
\]
where \( \hat{H}(t) \) is the Hamiltonian of the system. For generality, we have allowed for a possible explicit time-dependence in \( \hat{H} \). Now, a dynamical variable \( \hat{G}(t) \) with

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