Building on work by Fermat and Huygens, Hamilton transformed the study of geometrical optics in his very first paper, presented when still in his teens. His ‘characteristic function’ was an analytical way to describe wavefronts, and in his hands a powerful tool to look at families of rays rather than isolated ones. His prediction of internal and external conical refraction in some crystals and its spectacular verification in a few months established his reputation among his contemporaries. This formulation of optics uncovered many general properties, not easy to see in the conventional method of tracing individual rays. The deepest outcome of his early optical work was a parallel view of the mechanics of particles, which played a fundamental role in the birth of quantum mechanics and continues to be the standard framework for classical mechanics up to the present time.

Introduction

High school students are all exposed to geometrical optics – the reflection and refraction of rays of light according to the two well-known laws. These lead to analysis of mirrors, prisms, and lenses. While the experimental side of the subject has some charm – catching images on a screen in a darkened room, and peering at spectra through a telescope – the theory is rather uninspiring. One follows rules such as ‘draw a ray through the focus and let it emerge parallel to the axis’ or ‘draw a ray through the optic centre of the lens and let it pass undeviated’. Images are located by intersections of these
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This article introducing Hamilton’s work in its historical context is intended as an antidote to the early dose that most of us have received. It is one of the best kept secrets of theoretical physics that there is a better way of doing geometrical optics, nearly two hundred years old. The ten volume *Course of Theoretical Physics* by Landau and Lifshitz has all of four pages devoted to this in Volume 2, but they at least capture the essence!

**History**

We begin with the historical background. It was known from ancient times that light travels in a straight line in a uniform medium such as air or water. This is clearly the path of shortest length between two points. Already in the first century AD (now called CE, for ‘common era’). Heron, a Greek mathematician in Alexandria, stated that when light is reflected from a plane mirror, the usual law of reflection ensures that it takes the shortest path from source to receiver *via the mirror* (*Figure 1a*). (Mathematicians in those days were also engineers – Heron had an early version of a steam engine!) At a minimum, a first order change in the point of reflection C causes zero first order change in the path length (*Figure 1b*). (Usually, a function looks like a parabola near its minimum, so there is a second order change.) However, the reverse is not true. A zero first order change can also occur near a maximum, and reflection from a curved mirror provides an example (*Figure 1c*).

Fermat (1662) brought refraction into the same framework by suggesting that light travels more slowly in a medium like water, compared to air. This means that the path of least time is not the path of shortest length. It is advantageous to lengthen the portion of the path which is in air, so as to shorten the part in water where light travels more slowly. The speed of light in a medium of refractive index $n$ is $c/n$; so the formula for the time