Abstract In this paper, we give a new definition for the space of non-holomorphic Jacobi Maaß forms (denoted by $J_{k,m}^{nh}$) of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{N}$ as eigenfunctions of a degree three differential operator $\mathcal{C}_{k,m}$. We show that the three main examples of Jacobi forms known in the literature: holomorphic, skew-holomorphic and real-analytic Eisenstein series, are contained in $J_{k,m}^{nh}$. We construct new examples of cuspidal Jacobi Maaß forms $F_f$ of weight $k \in 2\mathbb{Z}$ and index 1 from weight $k - 1/2$ Maaß forms $f$ with respect to $\Gamma_0(4)$ and show that the map $f \mapsto F_f$ is Hecke equivariant. We also show that the above map is compatible with the well-known representation theory of the Jacobi group. In addition, we show that all of $J_{k,m}^{nh}$ can be “essentially” obtained from scalar or vector valued half integer weight Maaß forms.

Keywords Jacobi forms · Maass forms · Jacobi group · Automorphic representation

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1 Introduction

The theory of Jacobi forms has been studied extensively in the last few decades. One of the important features of Jacobi forms is that they form a bridge between the space of elliptic modular forms and Siegel modular forms. This fact is exploited to give a proof of the Saito-Kurokawa conjecture, which states that there is a lifting from elliptic modular forms to Siegel cusp forms of genus 2 (see [10]). In [13], Ikeda has used Jacobi forms of higher genus to prove a conjecture of Duke and Imamoglu (see [8]), which generalizes the Saito-Kurokawa conjecture. A very nice and systematic development of the theory of holomorphic Jacobi forms is given in the book [10] by Eichler and Zagier. In addition to the
holomorphic Jacobi forms, there are two main examples of Jacobi forms, namely, skew-holomorphic Jacobi forms and real analytic Eisenstein series. Analogous to Maaß forms on GL\(_2\), one would like to have a theory of non-holomorphic Jacobi forms. Such a theory should, at the least, include the above mentioned examples of Jacobi forms as subspaces and, more ambitiously, should account for all possible automorphic forms on the Jacobi group. A theory of non-holomorphic Jacobi forms is desirable for several reasons. Firstly, given an irreducible automorphic representation of the adelic points of the Jacobi group, we would be able to pick distinguished vectors in the representation and associate to them classical modular forms. Secondly, a theory of non-holomorphic Jacobi forms of higher genus will help obtain a lifting from representations of GL\(_2\), whose archimedean component is not a holomorphic discrete series, to representations of the symplectic group of higher genus (in analogy to Ikeda’s lift). Very little is known about the latter problem.

There have been a few attempts to define non-holomorphic Jacobi forms (see [5], [23]), but the theory developed so far is somewhat unsatisfactory. In this paper, we introduce a new way to define non-holomorphic Jacobi Maaß forms of weight \(k \in \mathbb{Z}\) and index \(m \in \mathbb{N}\).

- In [10], Eichler and Zagier define the holomorphic Jacobi forms \(F^h\) of weight \(k > 0\) and index \(m\) with respect to \(G^J(\mathbb{Z})\).
- Skoruppa defines the skew-holomorphic Jacobi forms \(F^{sh}\) of weight \(k > 0\) and index \(m\) in [20].
- In [1], Arakawa defines the real analytic Jacobi Eisenstein series \(E_{k,m}\) of weight \(k \in \mathbb{Z}\) and index \(m\), which is a generalization of the holomorphic Jacobi Eisenstein series from [10].

If \(F\) is any one of \(F^h, F^{sh}\) or \(E_{k,m}\), and is a Hecke eigenform as well, one can construct an irreducible automorphic representation \(\pi_F\) of \(G^J(\mathbb{A})\). If \(F\) is either \(F^h\) or \(F^{sh}\), then \(\pi_F\) is cuspidal and if \(F = E_{k,m}\) then \(\pi_F\) is not cuspidal.

Let \(\pi_F = \otimes \pi_F, p\). Then in each of the three cases the non-archimedean local representations are spherical and completely determined by the classical Hecke eigenvalues. On the other hand, the archimedean representations are completely different:

1. If \(F = F^h\) we see that \(\pi_{F,\infty}\) is a lowest weight discrete series representation \(\pi_{m,k}^+\) (see [4]),
2. If \(F = F^{sh}\) we see that \(\pi_{F,\infty}\) is a highest weight discrete series representation \(\pi_{m,k}^-\) (see [3]) and
3. If \(F = E_{k,m}\) we see that \(\pi_{F,\infty}\) is a principle series representation \(\pi_{2s-1,m,\pm 1}\) (see [5]).

The notations for the archimedean representations will be explained in details in Proposition 7.3. The type of archimedean representation obtained depends on the scalar with which the Casimir operator \(C\) (defined in (6)) of the Jacobi group acts on the representation \(\pi_{F,\infty}\). We pull back \(C\) to an operator \(C^{k,m}\) on functions on \(\mathcal{H} \times \mathbb{C}\), where \(\mathcal{H}\) is the complex upper half plane. Then the scalar used to determine \(\pi_{F,\infty}\) is precisely the eigenvalue of the operator \(C^{*,m}\) acting on \(F\), where \(\ast\) is the weight of \(F\). So, we see that the representation \(\pi_F\) is completely determined by the Hecke eigenvalues, the integers \(k, m\) and the eigenvalue of the differential operator \(C^{*,m}\) acting on \(F\).

The above discussion leads us to the conclusion that the most general notion of a Jacobi form with respect to \(G^J(\mathbb{Z})\) (which would include the three examples above) is a function, which is an eigenfunction of the degree 3 differential operator \(C^{k,m}\) and satisfies the automorphy condition with respect to the non-holomorphic automorphy factor defined in (2). We make this definition precise in Definition 3.2 and denote the space of Jacobi Maaß forms