Further generalizations of the Gale-Nikaido-Debreu theorem

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Abstract A fixed point theorem on compact compositions of acyclic maps on admissible (in the sense of Klee) convex subsets of a t.v.s. is applied to generalize Gwinner’s extensions of the Walras excess demand theorem and of the Gale-Nikaido-Debreu theorem.

Keywords Kakutani map · Acyclic map · Admissible set (in the sense of Klee) · Walras theorem · Gale-Nikaido-Debreu theorem

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1 Introduction

In 1981, Gwinner [8] displayed relations and connections between some of the most fundamental results of modern nonlinear analysis in the form of a circular tour. His tour starts in a traditional way, but also ends with the classical Knaster-Kuratowski-Mazurkiewicz (simply KKM) theorem [11]; thus eight results in [8] are in some wide sense equivalent to the KKM theorem. Nowadays there are nearly one hundred such equivalent results; see [21] and references therein. Especially, in [8], an infinite dimensional analogue of the Walras’ excess demand theorem was given, which is equivalent to the Fan-Glicksberg fixed point theorem.

On the other hand, in our previous works [19, 20], we applied a fixed point theorem for compact compositions of acyclic maps on admissible (in the sense of Klee)
convex subsets of a t.v.s. to obtain a cyclic coincidence theorem for acyclic maps, generalized von Neumann type intersection theorems, the Nash type equilibrium theorems, and the von Neumann minimax theorem.

In the present paper, our fixed point theorem is applied to generalize Gwinner’s extensions of the Walras excess demand theorem and of the Gale-Nikaido-Debreu theorem. A new generalization of the Walras theorem is shown to be equivalent to one of our fixed point theorems. We follow Gwinner’s method.

2 Preliminaries

All spaces are assumed to be Hausdorff and a t.v.s. means a topological vector space.

A multimap or map $F : X \rightarrow Y$ is a function from a set $X$ into the set $2^Y$ of nonempty subsets of $Y$; that is, a function with the values $F(x) \subseteq Y$ for $x \in X$ and the fibers $F^-(y) := \{ x \in X \mid y \in F(x) \}$ for $y \in Y$. For $A \subseteq X$, let $F(A) := \cup\{F(x) \mid x \in A\}$. For any $B \subseteq Y$, the (lower) inverse of $B$ under $F$ is defined by $F^-(B) := \{ x \in X \mid F(x) \cap B \neq \emptyset \}$.

For topological spaces $X$ and $Y$, a map $F : X \rightarrow Y$ is said to be closed if its graph

$$\text{Gr}(F) := \{(x, y) \mid y \in F(x), \ x \in X\}$$

is closed in $X \times Y$, and compact if $F(X)$ is contained in a compact subset of $Y$.

$F : X \rightarrow Y$ is said to be upper semicontinuous (u.s.c.) if, for each closed set $B \subseteq Y$, $F^-(B)$ is closed in $X$; lower semicontinuous (l.s.c.) if, for each open set $B \subseteq Y$, $F^-(B)$ is open in $X$; and continuous if it is u.s.c. and l.s.c.

If $F$ is u.s.c. with closed values and if $Y$ is regular, then $F$ is closed. The converse is true whenever $Y$ is compact.

Recall that a nonempty topological space is acyclic if all of its reduced Čech homology groups over rationals vanish. For nonempty subsets in a t.v.s., convex $\Rightarrow$ star-shaped $\Rightarrow$ contractible $\Rightarrow$ $\omega$-connected $\Rightarrow$ acyclic $\Rightarrow$ connected, and not conversely.

For topological spaces $X$ and $Y$, a map $F : X \rightarrow Y$ is called a Kakutani map whenever $Y$ is a convex subset of a t.v.s. and $F$ is u.s.c. with compact convex values; and an acyclic map whenever $F$ is u.s.c. with compact acyclic values.

Let $\nabla(X, Y)$ be the class of all acyclic maps $F : X \rightarrow Y$, and $\nabla_c(X, Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

The following is a particular form of our previous work [16, 17, 21, 22]:

**Theorem 1** Let $X$ be a nonempty convex subset of a locally convex t.v.s. $E$ and $T \in \nabla_c(X, X)$. If $T$ is compact, then $T$ has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

A nonempty subset $X$ of a t.v.s. $E$ is said to be admissible (in the sense of Klee) provided that, for every compact subset $K$ of $X$ and every neighborhood $V$ of the origin $0$ of $E$, there exists a continuous map $h : K \rightarrow X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace $L$ of $E$. 