Convergence theorems based on the shrinking projection method for variational inequality and equilibrium problems

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Abstract The purpose of this paper is to introduce a hybrid projection algorithm based on the shrinking projection method for two relatively weak nonexpansive mappings. We prove strong convergence theorem which approximate the common element in the fixed point set of two such mappings, the solution set of the variational inequality and the solution set of the equilibrium problem in the framework of Banach spaces. Our results improve and extend previous results.

Keywords Banach space · Fixed point · Projection · Relatively weak nonexpansive mapping · Shrinking projection method · Strong convergence

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1 Introduction

Let \( E \) be a Banach space and let \( E^* \) be the dual of \( E \) and let \( C \) be a closed and convex subset of \( E \). Let \( J \) be the normalized duality mapping from \( E \) into \( 2^{E^*} \) given by

\[
Jx = \{ x^* \in E^* : \langle x, x^* \rangle = \| x \| \| x^* \|, \| x \| = \| x^* \| \}, \forall x \in E,
\]

where \( \langle \cdot, \cdot \rangle \) is the generalized duality pairing between \( E \) and \( E^* \). It is well known that if \( E^* \) is uniformly convex, then \( J \) is uniformly continuous on bounded subsets of \( E \). Some properties of the duality mapping can be found in [9, 33, 39].

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Let $A : C \rightarrow E^*$ be an operator. We consider the following variational inequality:

Find $x \in C$, such that $\langle Ax, y - x \rangle \geq 0$, for all $y \in C$. \hfill (1.1)

A point $x_0 \in C$ is called a solution of the variational inequality (1.1) if for every $y \in C$, $\langle Ax_0, y - x_0 \rangle \geq 0$. The solution set of the variational inequality (1.1) is denoted by $VI(A, C)$.

If $C$ is a nonempty, closed and convex subset of a Hilbert space $H$ and $P_C : H \rightarrow C$ is the metric projection of $H$ onto $C$, then $P_C$ is nonexpansive, i.e., $\|P_Cx - P_Cy\| \leq \|x - y\|$, for all $x, y \in H$. This fact actually characterizes Hilbert spaces and, consequently, it is not available in more general Banach spaces. In this connection, Alber [1] recently introduced a generalized projection operator $\Pi_C$ in a Banach space $E$ which is an analogue of the metric projection in Hilbert spaces. Most recently, applying the generalized projection operator in uniformly convex and uniformly smooth Banach spaces, Li [19] established the following Mann type iterative scheme for solving variational inequalities without assuming the monotonicity of $A$ in compact subsets of Banach spaces: For any $x_0 \in C$, define a Mann type iteration scheme as follows

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - (Ax_n - \xi)), \quad n = 1, 2, 3, \ldots,$$

where $\{\alpha_n\}$ satisfies conditions $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$, $A : C \rightarrow E^*$ is a continuous mapping on a compact convex subset $C$ of $E$ such that

$$\langle Tx - \xi, J^*(Jx - (Ax - \xi)) \rangle \geq 0, \quad \text{for all } x \in C, \xi \in E^*.$$

It is proved in [19] that the variational inequality

$$\langle Ax - \xi, y - x \rangle \geq 0, \quad \forall y \in C$$

has a solution $x^* \in C$ and there exists a subsequence $\{n_i\} \subset \{n\}$ such that $\{x_{n_i}\}$ converges strongly to $x^*$ as $i \rightarrow \infty$. Moreover, Fan [13] established some existence results of solutions and the convergence of a Mann type iterative scheme for the variational inequality (1.1) in noncompact subsets of Banach spaces. More precisely, he proved the following theorem:

**Theorem Fan** (Fan [13], Theorem 3.3) Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a closed and convex subset of $E$. Suppose that there exists a positive number $\beta$, such that

$$\langle Ax, J^*(Jx - \beta Ax) \rangle \geq 0, \quad \text{for all } x \in C,$$

and $J - \beta A : C \rightarrow E^*$ is compact. If

$$\langle Ax, y \rangle \leq 0, \quad \text{for all } x \in K, \ y \in VI(A, C),$$

then the variational inequality (1.1) has a solution $x^* \in C$ and the sequence $\{x_n\}$ defined by the following iterative scheme:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \Pi_C(Jx_n - \beta Ax_n), \quad n = 1, 2, 3, \ldots,$$

has a solution $x^* \in C$ and there exists a subsequence $\{n_i\} \subset \{n\}$ such that $\{x_{n_i}\}$ converges strongly to $x^*$ as $i \rightarrow \infty$. Moreover, Fan [13] established some existence results of solutions and the convergence of a Mann type iterative scheme for the variational inequality (1.1) in noncompact subsets of Banach spaces. More precisely, he proved the following theorem: