Abstract In this paper we study smooth, non-special scrolls $S$ of degree $d$, genus $g \geq 0$, with general moduli. In particular, we study the scheme of unisecant curves of a given degree on $S$. Our approach is mostly based on degeneration techniques.

Keywords Ruled surfaces · Hilbert schemes of scrolls · Moduli · Embedded degenerations

Mathematics Subject Classification (2000) 14J26 · 14D06 · 14C20 · (Secondary) 14H60 · 14N10

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1 Introduction

It is well-known that the study of vector bundles on curves is equivalent to the one of scrolls in projective space. In the present article, we will mostly take the projective point of view, together with degeneration techniques, in order to study smooth, non-special scroll surfaces of degree $d$, sectional genus $g \geq 0$, with general moduli, which are linearly normal in $\mathbb{P}^R$, $R = d - 2g + 1$. However, we will bridge this approach with the vector-bundle one showing how projective geometry and degenerations can be used in order to improve some known results about rank-two vector bundles and to obtain some new ones (cf. also [4]).

The first three sections of the paper basically contain some folklore, which we think will be useful for a possible reader. In Sect. 2 and 3 we fix notation and terminology and recall preliminary results on scrolls. In Sect. 4 we introduce the vector bundle setting.

If $d \geq 2g + 3 + \min\{1, g - 1\}$, such scrolls fill up a unique component $H_{d, g}$ of the Hilbert scheme of surfaces in $\mathbb{P}^R$ which dominates $\mathcal{M}_g$ (this result is essentially contained in [2]; see [3] for an explicit statement and different proofs). Let $[S] \in H_{d, g}$ be a general point, such that $S \cong \mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a very ample rank-two vector bundle of degree $d$ on $C$, a curve of genus $g$ with general moduli, and $S$ is embedded in $\mathbb{P}^R$ via the global sections of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$. In Sect. 5, we first recall that if $g \geq 2$ and $S$ is general, then $\mathcal{F}$ is stable (in case $g = 1$ there is a slightly different result; cf. Theorem 2 and Remark 1. This result is contained in [2]. We give here a short, independent proof). This suggests that $H_{d, g}$ plays, in the projective geometry setting, a role analogous to the one of the moduli stack of semistable rank-two vector bundles of degree $d$ over $\mathcal{M}_g$. We discuss in Remarks 3 and 4 a few examples showing that $H_{d, g}$ contains points corresponding to unstable bundles as well as to strictly semistable products of the type $C \times \mathbb{P}^1$. We finish Sect. 5 by describing two constructions closely related to the ones in [3] (cf. Constructions 1 and 2) which prove that $H_{d, g}$ contains smooth points corresponding to some reducible scrolls. The results in [2] also imply that $H_{d, g}$ contains points corresponding to reducible scrolls. However, the ones that we need to consider in this paper are different from those in [2] and therefore we have to introduce them here. Note that in [3] one proves that $H_{d, g}$ contains points corresponding to surfaces which are reducible in suitable unions of planes, thus solving an old problem posed by G. Zappa (cf. [36]). These planar degenerations however are not used here.

In Sect. 6 we consider the scheme $\text{Div}^1_{S, m}$ parametrizing unisecant curves of given degree $m$ on $S$ (cf. Definition 6). By using degeneration arguments involving the above constructions, we prove Theorem 4, which says that $S$ is a general ruled surface in the sense of Ghione in [11], Definition 6.1, (cf. Definition 7 below). Namely