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L- and M-weak compactness of positive semi-compact operators

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Abstract. We present some necessary and sufficient conditions for positive semi-compact operators being L-weakly compact and M-weakly compact respectively.

Keywords Banach lattices · Semi-compact operator · M-weakly compact operator · L-weakly compact operator

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1 Introduction

Let $E$ and $F$ be Banach lattices, and $F^+$ be the positive cone of $F$. A continuous operator $T : E \to F$ is said to be semi-compact if for each $\epsilon > 0$, there exists some $u \in F^+$ such that $T(U) \subset [-u, u] + \epsilon V$ where $U$, $V$ denote the closed unit balls of $E$ and $F$, respectively. A continuous operator $T : E \to F$ is said to be M-weakly compact whenever $\lim \|Tx_n\| = 0$ holds for every norm
bounded disjoint sequence \( \{x_n\} \) of \( E \). A continuous operator \( T : E \to F \) is said to be \( L \)-weakly compact whenever \( \lim \|y_n\| = 0 \) holds for every disjoint sequence \( \{y_n\} \) in the solid hull of \( T(U) \) (please see [1] for details).

It was known that each \( M \)-weakly compact (\( L \)-weakly compact) operator is semi-compact (see Proposition 3.6.10 of [8] for details). However, a semi-compact operator is not necessarily \( M \)-weakly compact (\( L \)-weakly compact), since each compact operator may not be \( M \)-weakly compact (\( L \)-weakly compact). For example, the operator \( T : \ell_1 \to \ell_\infty \) defined by \( T(a_1, a_2, \cdots) = (\sum_{n=1}^\infty a_n, \sum_{n=1}^\infty a_n, \cdots) = (\sum_{n=1}^\infty a_n)(1, 1, 1, \cdots) \) is a compact operator, but it is neither \( M \)-weakly compact nor \( L \)-weakly compact.

Our objective in this paper is to present some necessary and sufficient conditions for positive semi-compact operators being \( L \)-weakly compact and \( M \)-weakly compact respectively.

We prefer to [1] and [8] for any unexplained terms from the theory of Banach lattices and operators.

2 L-weak compactness of positive semi-compact operators

We start with a characterization for positive semi-compact operators being \( L \)-weakly compact.

**Theorem 1** Let \( E \) and \( F \) be nonzero Banach lattices. Then each semi-compact operator \( T : E \to F \) is \( L \)-weakly compact if and only if the norm of \( F \) is order continuous.

**Proof** Suppose that \( F \) has order continuous norm and \( T : E \to F \) is a semi-compact operator. Let \( U \) and \( V \) denote the closed unit ball of \( E \) and \( F \), \( \{y_n\} \subset F_+ \) be a disjoint sequence in the solid hull of \( T(U) \). Pick a sequence \( \{x_n\} \subset E_+ \) with \( \|x_n\| \leq 1 \) and \( 0 \leq y_n \leq |T x_n| \) for all \( n \). As \( T \) is semi-compact, there exists a \( u \in F_+ \) satisfying \( ||(T x_n) - u|| < \epsilon \). The equality \( |T x_n| = |T x_n \cup u + (|T x_n| - u)^+| \) implies that \( |T x_n| \in \epsilon V + [0, u] \). For each \( n \), by Riesz’s decomposition property, there exist \( 0 \leq u_n \in \epsilon V \) and \( v_n \in [0, u] \) with \( y_n = u_n + v_n \).

Note that \( F \) has order continuous norm and \( \{v_n\} \) is an ordered bounded disjoint sequence, Theorem 12.13 of [1] implies that \( \lim \|v_n\| = 0 \). The inequality \( \|y_n\| \leq \|u_n\| + \|v_n\| \leq \epsilon + \|v_n\| \) implies \( \lim \sup \|y_n\| \leq \epsilon \), hence \( \lim \|y_n\| = 0 \).

On the other hand, suppose that the norm on \( F \) is not order continuous, there exists some \( y \in F \) and a disjoint sequence \( \{y_n\} \subset [0, y] \) which does not converge to zero in norm. For nonzero Banach lattice \( E \), \( E' \neq \{0\} \), choosing \( 0 < x' \in E' \), and define \( T : E \to F \) by \( T(x) = x'(x)y \) for each \( x \in E \). Clearly, \( T \) is semi-compact as it is compact (its rank is one). Hence, \( T \) is \( L \)-weakly compact. Note that \( \{y_n\} \) is a disjoint sequence in the solid hull of \( T \) ball(\( E \)) as \( 2^{-1} |x'||[0, y] \subset \text{sol}(T \text{ball}(E)) \), together with the \( L \)-weak compactness of \( T \) imply that \( \lim \|y_n\| = 0 \), which is a contradiction. \( \square \)