A Reilly Formula and Eigenvalue Estimates for Differential Forms

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Abstract We derive a Reilly-type formula for differential $p$-forms on a compact manifold with boundary and apply it to give a sharp lower bound of the spectrum of the Hodge Laplacian acting on differential forms of an embedded hypersurface of a Riemannian manifold. The equality case of our inequality gives rise to a number of rigidity results, when the geometry of the boundary has special properties and the domain is non-negatively curved. Finally, we also obtain, as a byproduct of our calculations, an upper bound of the first eigenvalue of the Hodge Laplacian when the ambient manifold supports non-trivial parallel forms.

Keywords Manifolds with boundary · Hodge Laplacian · Spectrum · Rigidity

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1 Introduction

Let $\Omega$ be a compact Riemannian manifold of dimension $n + 1$ with smooth boundary $\Sigma$, and $f$ a smooth function on $\Omega$. In [14], Reilly integrates the Bochner formula for the 1-form $df$ and rewrites the boundary terms in a clever way to obtain what is known in the literature as the Reilly formula for the function $f$. Reilly used it to
prove the Alexandrov theorem, but the Reilly formula turned out to be extremely useful, and has been successfully applied in many other contexts. For example, it was applied by Choi and Wang in [3] to obtain a lower bound for the first eigenvalue $\lambda_1(\Sigma)$ of the Laplacian on functions on embedded minimal hypersurfaces of the sphere, and by Xia in [19], who proved the following lower bound: if the domain $\Omega$ has non-negative Ricci curvature and its boundary $\Sigma$ is convex, with principal curvatures bounded below by $c > 0$, then:

$$\lambda_1(\Sigma) \geq nc^2. \quad (1)$$

Moreover, the equality holds if and only if $\Omega$ is a ball of radius $\frac{1}{c}$ in Euclidean space.

In this paper we integrate the Bochner-Weitzenböck formula to obtain a Reilly-type formula for differential $p$-forms (see Theorem 3), and we apply it to give a sharp lower bound of the spectrum of the Hodge Laplacian acting on differential forms of an embedded hypersurface $\Sigma$ bounding a compact manifold $\Omega$. This lower bound can be seen as a generalization of Xia’s estimate to differential forms. The equality case gives a number of rigidity results.

The Reilly formula has been generalized in other contexts, for example in spin geometry (see [8]); we hope that the present paper fills a gap in the literature and that other applications could be found.

Let us give a brief overview of the results of the paper.

1.1 The Main Estimate

Let us state the main lower bound in precise terms. Fix a point $x \in \Sigma$ and let $\eta_1(x), \ldots, \eta_n(x)$ be the principal curvatures of $\Sigma$ with respect to the inner unit normal. The $p$-curvatures are by definition all possible sums $\eta_{j_1}(x) + \cdots + \eta_{j_p}(x)$ for increasing indices $j_1, \ldots, j_p \in \{1, \ldots, n\}$. Arrange the sequence so that it is non-decreasing: $\eta_1(x) \leq \cdots \leq \eta_n(x)$; then the lowest $p$-curvature is $\sigma_p(x) = \eta_1(x) + \cdots + \eta_p(x)$, and we set:

$$\sigma_p(\Sigma) = \inf_{x \in \Sigma} \sigma_p(x).$$

We say that $\Sigma$ is $p$-convex if $\sigma_p(\Sigma) \geq 0$, that is, if all $p$-curvatures are non-negative. Note that 1-convex means convex (all principal curvatures are non-negative) and $n$-convex means that $\Sigma$ has non-negative mean curvature. It is easy to verify that, if $\sigma_p \geq 0$, then $\sigma_q \geq 0$ for all $q \geq p$ and, moreover, $\frac{\sigma_p}{p} \leq \frac{\sigma_q}{q}$.

Here is the main estimate of this paper.

**Theorem 1** Let $\Omega$ be a compact $(n + 1)$-dimensional Riemannian manifold with smooth boundary $\Sigma$, and let $1 \leq p \leq \frac{n+1}{2}$. Assume that $\Omega$ has a non-negative curvature operator and the $p$-curvatures of $\Sigma$ are bounded below by $\sigma_p(\Sigma) > 0$. Then:

$$\lambda'_{1,p}(\Sigma) \geq \sigma_p(\Sigma)\sigma_{n-p+1}(\Sigma),$$

where $\lambda'_{1,p}(\Sigma)$ is the first eigenvalue of the Hodge Laplacian acting on exact $p$-forms of $\Sigma$. The equality holds if and only if $\Omega$ is a Euclidean ball.