2-D Duffing Oscillator: Elliptic Functions from a Dynamical Systems Point of View

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Received: 6 October 2011 / Accepted: 14 May 2012 / Published online: 30 May 2012
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Abstract  K. Meyer has advocated for the study of elliptic functions and integrals from a dynamical systems point of view. Here, we follow his advice and we propose the bidimensional Hamiltonian Duffing oscillator as a model; it allows us to deal with the elliptic integral of third kind directly. Focusing on bounded trajectories we do a detailed analysis of the solutions in the three regions defined by the parameters. In our opinion, for the study of elliptic functions, the model presented here represents an alternative to the pendulum or the free rigid body systems.

Keywords Integrable systems · Hamiltonian systems · Duffing oscillator · Jacobi elliptic functions · Elliptic integrals · Elliptic integral of the third kind

“students who have sat through courses on differential and algebraic geometry […] turned out be acquainted neither with the Riemann surface of an elliptic curve […] not even mentioning elliptic integrals of first kind […] They do not know that an elliptic integral expresses the time of motion along an elliptic phase

In Honor of Ken R. Meyer.

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curve in the corresponding Hamiltonian system.” On teaching mathematics, V.I. Arnold

“The three jewels whose effulgence has most dazzled me are Maxwell, Einstein and elliptic functions theories. [...] I have derived the greatest pleasure is the present work on elliptic functions. How enviable are Jacobi and Weierstrass to have been the creators of such a work of art!” Elliptic Functions and Applications, D. Lawden

1 Introduction

Along his academic life Ken Meyer has proved that both research and teaching may go together enriching each other. A case in point is the way in which he has presented Jacobi elliptic functions in his books and papers, considering simple dynamical models in order to illustrate several aspects of them. His way to handle with them has been a source of motivation for us on how to work with those functions; in our research we find them quite often. As the reader ought to have noticed, in this paper written in his honor, we have borrowed part of the title from one of him. In fact we vertebrate our contribution inspired by two of his papers [25,27] and some of the references provided there.

We consider a system which requires to work with the elliptic integral of third kind. More precisely, the aim of this paper is twofold. On the one hand the present contribution wishes to invite the reader to go again to The Amer. Math. Monthly paper [25]; it is a pleasure to read it. On the other hand we add something we do not find there. Indeed, meanwhile the Legendre first kind elliptic integral is the only one considered by Meyer, the 2D Duffing system, through one of the quadratures, allows to introduce directly the third elliptic integral. But, in that respect we have to say that Meyer did not lead alone the reader eager to enter in that study. His short but select list of references plays a key role, paving the way in the extensive and impressive literature about these functions; one of them we have found very convenient for the study of the third kind elliptic integral has been Greenhill [15]. When we pair Greenhill with the fine work of Lawden [20], almost a century apart, and more theoretically oriented, this leads finally to grasp some of the beauty encapsulated behind the formulas for dealing with this integral and associated elliptic functions.

Duffing system is ubiquitous in publications of Meyer, when he illustrates methods and concepts. But, like in other classic examples, with Duffing occurs that there are different systems in the literature under the same name. By 1D-Duffing we mean the Hamiltonian system defined by the function

\[ H(x, X; \omega, \alpha) = \frac{1}{2}X^2 + \frac{\omega}{2}x^2 + \frac{\alpha}{4}x^4. \]  

(1)

By extension, we study here the differential system given by the Hamiltonian function

\[ H(x, y, X, Y; \omega, \alpha) = \frac{1}{2}(X^2 + Y^2) + \frac{\omega}{2}(x^2 + y^2) + \frac{\alpha}{4}(x^2 + y^2)^2 \]  

(2)

and we will call it bidimensional Hamiltonian Duffing system (2DHD); other authors refer to it as nonlinearly coupled Duffing equations [33]; its solution is given by elliptic