Linear Stability of the \( n \)-gon Relative Equilibria of the \((1 + n)\)-Body Problem

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Abstract We consider the linear stability of the regular \( n \)-gon relative equilibria of the \((1 + n)\)-body problem. It is shown that there exist at most two kinds of infinitesimal bodies arranged alternatively at the vertices of a regular \( n \)-gon when \( n \) is even, and only one set of identical infinitesimal bodies when \( n \) is odd. In the case of \( n \) even, the regular \( n \)-gon relative equilibrium is shown to be linearly stable when \( n \geq 14 \). In each case of \( n = 8, 10 \) and 12, linear stability can also be preserved if the ratio of two kinds of masses belongs to an open interval. When \( n \) is odd, the related conclusion on the linear stability is recalled.

Keywords \((1 + n)\)-body · Linear stability · Relative equilibrium

1 Introduction

The famous essay Maxwell [1] discussed the stability of the rings of Saturn by mathematical modeling. Under the hypothesis that the motion of the rings is uniform and should be stable, he abandoned the possibilities that the rings are connected solid, or continuous liquid. He concluded that the rings should be composed of countless discrete particles ([1], p. 66) after an important intermediate study. In that study, he
conceived a very nearly circular and uniform ring of satellites with equal masses, and found that the ring is linearly stable provided the ratio of Saturn’s mass to the total mass of \( n \) satellites is greater than 0.4352\( n^2 \) ([1], p. 25 or p. 59). This essay has aroused attentions of many aspects. One can also refer to a French monument written by Tisserand [2].

Pendse [3] pointed out that Maxwell’s ring model could not be applied to the cases when \( n \) is smaller than seven, because Maxwell assumed tacitly the center of primary to be the center of masses. Scheeres and Vinh [4] accepted Pendse’s argument, and analyzed the characteristic equations very attentively. They pointed out a miscalculation of Pendse, and concluded that the upper bound of the ratio of the total mass of \( n \) satellites to the dominant mass for stability can be represented by an asymptotic series of \( n \) if \( n \geq 7 \), and is about \( 1/(0.4352n^2) \) when \( n \) is large. Similar results are also achieved by Roberts [5] and Vanderbei and Kolemen [6].

Consider the planar \( N \)-body problem with \( N = n + 1 \) in a uniformly rotating coordinate system with the origin at the center of masses. Suppose one mass is large and the other \( n \) masses are very small. As the order of magnitude of the \( n \) small masses tends to zero, the limiting case of a relative equilibrium of this problem, is called a relative equilibrium of the (1 + \( n \))-body problem, which was defined by Hall in an unpublished paper [7]. The relationship between the relative equilibria of (1 + \( n \))-body problem and those of the \( N \)-body problem with \( N = n + 1 \) has been revealed by Moeckel [8], and will be mentioned in Sect. 2.

If all the infinitesimal masses are equal, Casasayas et al. [9] showed that when \( n > e^{73} \), the only stationary configuration is Maxwell’s ring configuration. When \( n \leq 8 \), other configurations exist, referring to Salo and Yoder [10]. According to Salo and Yoder’s numerical quest, it seems that there is only one type of stationary configuration when \( n \geq 9 \), and this conjecture is also supported by the numerical experiment of Carles Simó ([11], p. 326). When 2 \( \leq n \leq 4 \), the numerical results on the numbers of stationary configurations also coincide with analytical proofs, see Cors et al. [11], Albouy and Fu [12]. For the cases of \( n = 5, 6, 7, 8, 9 \), they seem to be much more difficult to deal with, as the case of \( n = 4 \) had already been claimed by Hall to be difficult to handle ([7], p. 12).

If the infinitesimal masses are arbitrarily given, Maxwell said that “we must calculate the disturbing forces due to any given displacement of the ring”([1], p. 38), so it is difficult to determine the stability of a non-regular configuration. Salo and Yoder [10] studied a special case of \( n = 3 \) with a background of Saturn’s coorbital satellites, and Renner and Sicardy [13] studied the cases of \( n = 3, 4, 5 \). Recently, Corbera et al. [14] researched into the case of \( n = 3 \) and obtained two classes of new configurations generated by changing the infinitesimal masses. They also calculated the number of configurations with a high precision, and found that the number varies from five to seven.

The new results gotten in this paper are especially about the existence and linear stability of the regular \( n \)-gon relative equilibria in the (1 + \( n \))-body problem with two kinds of infinitesimal masses. In Sect. 2, we introduce Hall’s potential function on the regular \( n \)-gon relative equilibria of the (1 + \( n \))-body problem. Then the method for calculating eigenvalues of the Hessian matrix of Hall’s potential function is described in Sects. 2.2 and 2.3. In Sect. 3, we show the existence of a set of positive infinitesimal