Homogenization of an Optimal Control Problem with State-constraints

S. Kesavan · T. Muthukumar

Abstract  Homogenization of an optimal control problem, whose state equations and cost functionals involve rapidly oscillating coefficients, with constraints on state is studied in the framework of $\Gamma$-convergence. The problem is considered in both perforated and non-perforated domains. The case where the cost of the control is of the order of the parameter is also considered.

Keywords  Homogenization · Optimal control · $\Gamma$-convergence · State-constraints

Introduction

This paper is devoted to the study of homogenization of an optimal control problem with constraints on state whose state equation (given by a second order elliptic boundary value problem) and cost functional (involving a Dirichlet-type integral of the state) have rapidly oscillating coefficients. We consider both the perforated and non-perforated cases of the problem. The crucial part of the problem is to identify the limit admissible set of the homogenized problem. This is established and it turns out to be an application of the perturbed limit matrix obtained by Kesavan and Saint Jean Paulin in [4], for the non-perforated case and in [5], for the perforated case. We also consider the cases where the cost of the control is (a) dependent on the parameter (cheap control) and (b) independent of the parameter.

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Let $0 < a < b$, $0 < c < d$, $N > 0$ be given constants. We denote by $\mathcal{M}(a, b, \Omega)$ the set of all $n \times n$ matrices, $A = A(x)$, whose entries are in $L^\infty(\Omega)$ such that,

$$a|\xi|^2 \leq A(x)\xi.\xi \leq b|\xi|^2 \quad a.e. \, x \quad \forall \xi = (\xi_i) \in \mathbb{R}^n.$$
Let $A \in \mathcal{M}(a, b, \Omega)$ and $B \in \mathcal{M}(c, d, \Omega)$ with $B$ symmetric. Let $U$ be a closed convex subset of $L^2(\Omega)$ and let $f \in L^2(\Omega)$ be a given function. The basic optimal control problem is the following: Find $\theta^* \in U$ such that,

$$J(\theta^*) = \min_{\theta \in U} J(\theta),$$

where the cost functional, $J(\theta)$, is defined by

$$J(\theta) = \frac{1}{2} \int_\Omega B \nabla u \cdot \nabla u \, dx + \frac{N}{2} \| \theta \|_2^2$$

(1.1)

and the state $u = u(\theta)$ is the weak solution in $H^1_0(\Omega)$ of the boundary value problem

$$\begin{cases}
-\text{div}(A \nabla u) = f + \theta & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

(1.2)

It can be shown by direct methods in the calculus of variations that there is a unique optimal control, $\theta^* \in U$, minimizing $J$ over $U$ (cf. [2, Theorem 1.15 and Prop. 1.20]).

The situation in which the matrices $A$ and $B$ above depend on $\epsilon$, a parameter which tends to zero, has been studied by Kesavan and Vanninathan (cf. [7]) in the periodic case, and by Kesavan and Saint Jean Paulin in non-perforated (cf. [4]) and perforated domains (cf. [5]). In these papers it is shown that, in the limit, the optimal controls $\theta^*_\epsilon$ converged to $\theta^*$ which turns out to be an optimal control of a problem of the type (1.1)–(1.2) where the matrix $A$ is none other than the $H$-limit ($H_0$-limit, for perforated domains) of $\{A_\epsilon\}$ while the matrix $B = B^*$ is a perturbation of the $H$-limit ($H_0$-limit, for perforated domains) of $\{B_\epsilon\}$. The matrix $B^*$ is symmetric and positive definite (cf. [4]). However, in the non-perforated case the set $U$ was assumed to be independent of $\epsilon$ and in the perforated case obstacle type sets were considered.

The case where the cost of the control (cf. $N$ in (1.1)) is of the order of $\epsilon$ was considered by Kesavan and Saint Jean Paulin (cf. [6]) when the admissible set was either $L^2(\Omega)$ (unconstrained case) or the positive cone.

A useful role of $B^*$ in homogenization theory was observed by Kesavan and Rajesh (cf. [3]). Given $g_\epsilon \to g$ strongly in $H^{-1}(\Omega)$, if $v_\epsilon \in H^1_0(\Omega)$ is the solution of

$$\begin{cases}
-\text{div}(A_\epsilon \nabla v_\epsilon) = g_\epsilon & \text{in } \Omega \\
v_\epsilon = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.3)

then $v_\epsilon$ converges weakly to $v$ in $H^1_0(\Omega)$ where $v$ solves the homogenized problem,

$$\begin{cases}
-\text{div}(A_0 \nabla v) = g & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.4)

$A_0$ being the $H$-limit of $\{A_\epsilon\}$.

For these states, it was observed that

$$\int_\Omega B_\epsilon \nabla v_\epsilon \cdot \nabla v_\epsilon \, dx \to \int_\Omega B^* \nabla v \cdot \nabla v \, dx.$$ 

(1.5)

It was also observed that $B^*$ is the limit, in the sense of distributions, of $^t D_\epsilon B_\epsilon D_\epsilon$, i.e., for $1 \leq i, j \leq n$,

$$^t D_\epsilon B_\epsilon D_\epsilon e_i.e_j \to B^* e_i.e_j \text{ in } \mathcal{D}'(\Omega).$$

(1.6)