Coupled Nonlinear Oscillators: Metamorphoses of Resonance Curves
The General Case of the Approximate Effective Equation

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Abstract We study dynamics of two coupled periodically driven oscillators. Periodic solutions of the approximate effective equation derived in our earlier work are determined within the Krylov–Bogoliubov–Mitropolsky approach used to compute the amplitude profiles. In the present paper we investigate metamorphoses of an amplitude profile induced by changes of the control parameters near a singular point of this function. It follows that dynamics can be controlled in the neighbourhood of a singular point.

Keywords Coupled oscillators · Resonance curves · Singular points

Introduction

In the present paper we analyse dynamics of two coupled oscillators, one of which is driven by an external periodic force. Equations describing dynamics of such system are of form:

\[
\begin{align*}
    m_1 \ddot{x}_1 - V_1(\dot{x}_1) - R_1(x_1) + V_2(\dot{x}_2 - \dot{x}_1) + R_2(x_2 - x_1) &= f \cos(\omega t) \\
    m_2 \ddot{x}_2 - V_2(\dot{x}_2 - \dot{x}_1) - R_2(x_2 - x_1) &= 0
\end{align*}
\]

(1)

where \(V_1, R_1\) and \(V_2, R_2\) represent (nonlinear) force of internal friction and (nonlinear) elastic restoring force for mass \(m_1\) and mass \(m_2\), respectively. Important example of a mechanical model described by Eq. (1) is a dynamic vibration absorber which consists of a mass \(m_2\), attached to the primary vibrating system of mass \(m_1\) [1,2].
We shall consider a simplified model:

\[ R_1 (x_1) = -\alpha_1 x_1, \quad V_1 (\dot{x}_1) = -\nu_1 \dot{x}_1. \]  (2)

Dynamics of coupled periodically driven oscillators is very complicated [3–7]. We simplified the set equations (1), (2) deriving the exact fourth-order nonlinear equation for internal motion as well as approximate second-order effective equation in [8]. Applying the Krylov–Bogoliubov–Mitropolsky (KBM) method to these equations we have computed the corresponding nonlinear resonances (cf. [8] for the case of the effective equation). Dependence of the amplitude \( A \) of nonlinear resonances on the frequency \( \omega \) is much more complicated than in the case of Duffing oscillator and hence new nonlinear phenomena are possible. In the present paper we study metamorphoses of the function \( A(\omega) \) induced by changes of the control parameters.

The paper is organized as follows. In the next Section the exact 4th-order equation for the internal motion and approximate 2nd-order effective equations in non-dimensional form are presented. In Sect. 3 metamorphoses of amplitude profiles determined within the KBM approach for the approximate 2nd-order effective equation are defined and theory of algebraic curves is used to compute singular points on effective equation amplitude profiles — metamorphoses of amplitude profiles occur in neighbourhoods of such points. In Sect. 4 examples of analytical and numerical computations are presented for the effective equation. Our results are summarized in the last Section.

**Exact Equation for Internal Motion and its Approximations**

In new variables, \( x \equiv x_1, y \equiv x_2 - x_1 \), Eqs. (1), (2) can be written as:

\[
\begin{align*}
M \ddot{x} + \nu \dot{x} + \alpha x + V_e (\dot{y}) + R_e (y) &= f \cos (\omega t) \\
M_e (\ddot{y} + \dot{y}) - V_e (\dot{y}) - R_e (y) &= 0
\end{align*}
\]  (3)

where \( m \equiv m_1, m_e \equiv m_2, \nu \equiv \nu_1, \alpha \equiv \alpha_1, V_e \equiv V_2, R_e \equiv R_2 \).

Adding Eq. (3) we obtain important relation between variables \( x \) and \( y \):

\[
M \ddot{x} + \nu_1 \dot{x} + \alpha_1 x + m_e \ddot{y} = f \cos (\omega t),
\]  (4)

where \( M = m + m_e \).

We can eliminate variable \( x \) in (3) to obtain the following exact equation for relative motion:

\[
\left( M \frac{d^2}{dt^2} + \nu \frac{d}{dt} + \alpha \right) (\mu \ddot{y} - V_e (\dot{y}) - R_e (y)) + \epsilon m_e (\nu \frac{d}{dt} + \alpha) \ddot{y} = F \cos (\omega t),
\]  (5)

where \( F = m_e \omega^2 f, \mu = mm_e / M \) and \( \epsilon = m_e / M \) is a nondimensional parameter [8], see also Ref. [9] where separation of variables for a more general system of coupled equations was described. Equations (5), (4) are equivalent to the initial equations (1), (2).

For small \( \epsilon \) we can reject the term proportional to \( \epsilon \) to obtain the approximate equation which can be integrated partly to yield the effective equation (in the present paper we consider more general case than studied in [8] where \( \lambda_e = 0 \) was assumed):

\[
\mu \ddot{y} + v_e \dot{y} - \lambda_e \dot{y}^3 + \alpha_e y + \gamma_e y^3 = \frac{-m_e \omega^2 f}{\sqrt{M^2 (\omega^2 - \frac{\alpha}{2})^2 + v^2 \omega^2}} \cos (\omega t + \delta),
\]  (6)

where transient states were neglected and we assumed:

\[
R_e (y) = -\alpha_e y - \gamma_e y^3, \quad V_e (\dot{y}) = -v_e \frac{dy}{dt} + \lambda_e \dot{y}^3.
\]  (7)