Spatiospectral concentration in the Cartesian plane

Frederik J. Simons · Dong V. Wang

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Abstract We pose and solve the analogue of Slepian’s time–frequency concentration problem in the two-dimensional plane, for applications in the natural sciences. We determine an orthogonal family of strictly bandlimited functions that are optimally concentrated within a closed region of the plane, or, alternatively, of strictly spaciallimited functions that are optimally concentrated in the Fourier domain. The Cartesian Slepian functions can be found by solving a Fredholm integral equation whose associated eigenvalues are a measure of the spatiotemporal concentration. Both the spatial and spectral regions of concentration can, in principle, have arbitrary geometry. However, for practical applications of signal representation or spectral analysis such as exist in geophysics or astronomy, in physical space irregular shapes, and in spectral space symmetric domains will usually be preferred. When the concentration domains are circularly symmetric in both spaces, the Slepian functions are also eigenfunctions of a Sturm–Liouville operator, leading to special algorithms for this case, as is well known. Much like their one-dimensional and spherical counterparts with which we discuss them in a common framework, a basis of functions that are simultaneously spatially and spectrally localized on arbitrary Cartesian domains will be of great utility in many scientific disciplines, but especially in the geosciences.

Keywords Bandlimited function · Commuting differential operator · Concentration problem · Eigenvalue problem · Spectral analysis · Reproducing kernel · Spherical harmonics · Sturm–Liouville problem

F. J. Simons (✉) · D. V. Wang
Department of Geosciences, Princeton University, Princeton, NJ 08544, USA
e-mail: fjsimons@alum.mit.edu

Present Address:
D. V. Wang
Department of Statistics and Operations Research,
The University of North Carolina at Chapel Hill, Chapel Hill, NC 27599, USA
1 Introduction

The one-dimensional prolate spheroidal wave functions (pswf) have enjoyed an enduring popularity in the signal processing community ever since their introduction in the early 1960s (Landau and Pollak 1961, 1962; Slepian and Pollak 1961). Indeed, in many scientific and engineering disciplines the pswf and their relatives the discrete prolate spheroidal sequences (dpss; Grünbaum 1981; Slepian 1978) have by now become the preferred data windows to regularize the quadratic inverse problem of power spectral estimation from time-series observations of finite extent (Percival and Walden 1993).

At the deliberate cost of introducing spectral bias, windowing the data with a set of such orthogonal “tapers” lowers the variance of the “multitaper” average (Thomson 1982), which results in estimates of the power spectral density that are low-error in the mean-squared sense (Thomson 1990). As a basis for function representation, approximation and interpolation (Delsarte et al. 1985; Moore and Cada 2004; Shkolnisky et al. 2006; Xiao et al. 2001), or in stochastic linear inverse problems (Bertero et al. 1985a,b; de Villiers et al. 2001; Wingham 1992), the pswf have been less in the public eye, especially compared to wavelet analysis (Daubechies 1992; Percival and Walden 2006), though, due to advances in computation, there has been a resurgent interest in recent years (Beylkin and Monzón 2002; Karoui and Moumni 2008; Khare and George 2003; Walter and Soleski 2005), in particular as relates to using them for the numerical solution of partial differential equations (Beylkin and Sandberg 2005; Boyd 2003, 2004; Chen et al. 2005).

The pswf are the solutions to what has come to be known as “the concentration problem” (Flandrin 1998; Percival and Walden 1993) of Slepian, Landau and Pollak, in which the energy of a bandlimited function is maximized by quadratic optimization inside a certain interval of time. Vice versa, it refers to the maximization of the spectral localization of a timelimited function inside a certain target bandwidth (Slepian 1983). In the first version of the problem, the bandlimited, time-concentrated pswf form an orthogonal basis for the entire space of bandlimited signals that is also orthogonal over the particular time interval of interest. In the second version the timelimited, band-concentrated pswf are a basis for square-integrable broadband signals that are exactly confined to the interval (Landau and Pollak 1961; Slepian and Pollak 1961). In general we shall refer to all singular functions of time-bandwidth or space-bandwidth projection operators as “Slepian functions”.

The fixed prescription of the “region of interest” in physical or spectral space is a deliberately narrow point of view that is well suited to scientific or engineering studies where the assumption of stationarity, prior information, or the availability of data will dictate the interval of study from the outset. This distinguishes the Slepian functions philosophically from the eigenfunctions of full-phase-space localization operators (Daubechies 1988; Simons et al. 2003) or wavelets (Daubechies and Paul 1988; Olhede and Walden 2002), with which they nevertheless share strong connections (Lilly and Park 1995; Shepp and Zhang 2000; Walter and Shen 2004, 2005). Strict localization of this type remained the driving force behind the development of Slepian