ANALOGUES OF MIYACHI, COWLING-PRICE AND MORGAN THEOREMS FOR COMPACT EXTENSIONS OF $\mathbb{R}^n$\footnote{This work was completed with the support of D.G.R.S.R.T through the Research unit 00 UR 1501}

Salma Azaouzi, Ali Baklouti and Mounir Elloumi

Sfax University, Department of Mathematics, Faculty of Sciences at Sfax,
Route de Soukra, 3038, Sfax, Tunisia

e-mails: Ali.Baklouti@fss.rnu.tn, azaouzisalma@yahoo.com,
mounir.elloumi@yahoo.fr

(Received 18 September 2012; accepted 21 December 2012)

Let $K$ be a compact subgroup of automorphisms of $\mathbb{R}^n$. We formulate and prove an analogue of Miyachi’s theorem for the semi-direct product $K \ltimes \mathbb{R}^n$. This allows us to solve the sharpness problems in the theorem of Cowling-Price and in the $L^p - L^q$ analogue of Morgan theorem for any compact extension of $\mathbb{R}^n$. These upshots are proved using the representations theory and the Plancherel formula for the group Fourier transform.

**Key words**: Uncertainty principle; Fourier transform; Plancherel formula.

1. **Introduction**

A classical version of the uncertainty principle, roughly speaking, states that no non-zero integrable function $f$ on $\mathbb{R}^n$, equipped with the natural inner product $\langle \cdot, \cdot \rangle$,
and its Fourier transform $\hat{f}$, defined by

$$\hat{f}(y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) \exp(-i\langle x, y \rangle) dx$$

cannot both be sharply localized. More precisely, it is not possible for an integrable function and its Fourier transform both to satisfy Gaussian estimates in such a way that the product of the variances of these Gaussian is strictly superior to one quarter. To make these principles concrete, Hardy’s theorem is a neat argumentative explanation (see [4], [10], [12] and [13]). Several generalizations of Hardy’s uncertainty principle have appeared since; the most notable among them being the result of Cowling-Price, which proves that there is no nontrivial function $f$ defined on $\mathbb{R}^n$ such that $f$ and its Fourier transform $\hat{f}$ can both decay very rapidly and consists in replacing Gaussian bounds on $f$ by Gaussian bounds in the $L^p$ sense and in the $L^q$ sense for $\hat{f}$ as well, where $1 \leq p, q \leq +\infty$ are real numbers. We have the following (cf. [3]):

**Theorem 1.1** — Let $1 \leq p, q \leq +\infty$ such that $\min(p, q)$ is finite, and let $\alpha, \beta$ be positive constants. Suppose that $f$ is a measurable function on $\mathbb{R}^n$ satisfying:

(i) $\|e^{x\|x\|^2} f\|_p < +\infty$,

(ii) $\|e^{\beta \|y\|^2} \hat{f}\|_q < +\infty$.

Then $f = 0$ almost everywhere whenever $\alpha \beta \geq \frac{1}{4}$. If $\alpha \beta < \frac{1}{4}$, then there exist infinitely many linearly independent functions satisfying (i) and (ii).

With the purpose of obtaining an even more general variant of Hardy’s uncertainty principle, Morgan’s theorem involved the generalized Gaussians. Combining the last two principles, Ben Farah and Mokni produced in [2] a version referred to as the $L^p - L^q$ Morgan uncertainty principle that has the advantage of unifying the previously mentioned results. They proved the following theorem.

**Theorem 1.2** — Let $p' > 2$, and let $q'$ be its conjugate. Let $1 \leq p, q \leq +\infty$, and let $\alpha, \beta$ be positive constants. Suppose that $f$ is a measurable function on $\mathbb{R}^n$ satisfying: