MAPS TO WEIGHT SPACE IN HIDA FAMILIES

Ravi Ramakrishna

Department of Mathematics, Cornell University, Ithaca, NY 14853, USA

e-mail: ravi@math.cornell.edu

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Abstract. Let \( \bar{\rho} \) be a two-dimensional \( \mathbb{F}_p \)-valued representation of the absolute Galois group of the rationals. Suppose \( \bar{\rho} \) is odd, absolutely irreducible and ordinary at \( p \). Then we show that \( \bar{\rho} \) arises from the irreducible component of a Hida family (of necessarily greater level than that of \( \bar{\rho} \)) whose map to weight space, including conjugate forms, has degree at least 4.

Key words: Galois representation; modular form; Hida theory.

1. Introduction

Let \( p \geq 5 \) be an odd prime and \( f \in S_2(\Gamma_0(Np)) \) a weight two eigenform that is new of level \( Np \) where \( p \nmid N \). Let \( \rho_f \) and \( \bar{\rho}_f \) be the \( p \)-adic and mod \( p \) representations of \( G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) associated to \( f \). We assume \( \bar{\rho}_f \) is absolutely irreducible so it is well-defined. As \( p \) is in the level of \( f \), the eigenvalue of \( U_p \) is \( \pm 1 \) and \( \rho_f \) is ordinary at \( p \), in particular \( \rho_f|_{G_p=\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} = \begin{pmatrix} \psi \epsilon & * \\ 0 & \psi^{-1} \end{pmatrix} \) where \( \epsilon \) is the \( p \)-adic cyclotomic character and \( \psi \) is unramified of order 1 or 2. We know that \( f \) belongs to a Hida family, by which in this paper we mean the irreducible component (of the spectrum) of the ordinary (arbitrary weight) Hecke algebra of tame level \( N \) containing \( f \). We will abuse the term ‘Hida family’ to refer to both the ring \( \mathbb{T} \) and \( T = \text{Spec}(\mathbb{T}) \). When we say the Hida family contains a point, we are referring to a map \( \text{Spec}(R) \to T \) for a suitable extension \( R \) of \( \mathbb{Z}_p \). Dimitrov and Ghate refer to the collection of all components at a fixed tame

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level having residual global representation $\bar{\rho}_f$ as its *Hida community*. It is well-known that the Hida family containing $f$ is a finite flat $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]] \simeq \mathbb{Z}_p[[T]]$-algebra $T$ that is an integral domain possessing homomorphisms $T \to \mathbb{Z}_p$ that correspond to classical eigenforms of weights $k \geq 2$ (and sometimes weight $k = 1$). If the reader prefers, rather than $f$, she may keep in mind the example of an elliptic curve with semistable reduction at $p$.

While it is known that $f$ determines $T$, little is known about how to recover the explicit information of the family $T$ from $f = \sum_{n=1}^{\infty} a_n q^n$. C. Franc has pointed out that given two eigenforms $g$ and $h$ of level $M$ that are congruent mod $p$, there seems to be no known algorithm to determine whether $g$ and $h$ are in the same family!

For $f$ with split multiplicative reduction, the $p$-adic $L$-function $L_p(f,s)$ has a trivial zero at $s = 1$. In the elliptic curve case Mazur, Tate and Teitelbaum conjectured a relation between the classical $L$-function $L(E,1)$ and $L'_p(E,1)$. Recall the $L$-invariant of a semistable at $p$ elliptic curve $E$ with Tate period $q_E$ is

$$L_E = \frac{\log q_E}{v_p(q_E)}.$$  

In [MTT] it was conjectured

$$L_p(E,1) = L_E \frac{L(E,1)}{\Omega_E}$$

where $\Omega_E$ is the real period of $E$. Greenberg and Stevens proved this conjecture in [GS] using Hida theory and by relating $L_E$ to the derivative of the $Frob_p$-eigenvalue in the Hida family. Indeed, their result applies to $f$ split multiplicative but in this more general case computing $L_f$ explicitly is very involved. See for instance [CST]. The $L$-invariant contains other interesting information. In [GS2] Greenberg and Stevens gave a simple criterion in terms of $L_f$ that guarantees the existence of another weight 2 point in $T$ whose level is prime to $p$. Below is a slight generalization of their result, Proposition 5.1 of [GS2].

**Theorem 1.** *(Greenberg-Stevens)* Let $p \geq 5$ be a prime and $f \in S_2(\Gamma_0(Np))$ have multiplicative reduction at $p$. Suppose $\bar{\rho}_f$ is absolutely irreducible as a $G_{\mathbb{Q}}$-module and that $v_p(L_f) < 1$. Then the Hida family $T$ containing $f$ contains another weight 2 form of level $N$ prime to $p$. In particular the map $\Lambda = \mathbb{Z}_p[[T]] \to T$ is not an isomorphism.

The differences between the result of [GS2] and Theorem 1 above are that we state the result along a particular Hida family (irreducible component), work with modular forms