BALANCING DIRICHLET SERIES AND RELATED L-FUNCTIONS

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We study the lacunary Dirichlet series obtained from the reciprocals of $s^{th}$ powers of balancing numbers. This function admits an analytic continuation to the entire complex plane. The series converges to irrational numbers at odd negative integral arguments. Finally, we also study the analytic continuation of the balancing $L$-function.

Key words: Balancing numbers; Dirichlet series; $L$-function.

1. INTRODUCTION

A series of the form $\sum_{n=1}^{\infty} a_n n^{-s}$ where $a_n, n \in \mathbb{N}$ is a complex sequence and $s$ is a complex number is called a Dirichlet series. Functions defined by this series very often relate algebraic properties in analytic terms. This mostly happens when $a_n$ is a multiplicative functions such as number of divisors of $n$, sum of divisors of $n$ or the Möbius function and so on. When $a_n = 1$ and $\text{Re}(s) > 1$, one gets $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ which is the Riemann zeta function, extensively available in literature [2, 4]. $\zeta(s)$ can be analytically continued to the whole complex plane, with only one simple pole at $s = 1$. Also, $\zeta(s)$ has an important symmetry around the line $\text{Re}(s) = \sigma = 1/2$, in the form of a functional equation. The trivial zeros of $\zeta(s)$ are located at $-2, -4, -6, \ldots$ and its values at negative odd integers are rational, and in fact, given by the Bernoulli numbers [9].

The series $\zeta_F(s) = \sum_{n=1}^{\infty} F_n^{-s}$ where $F_n$ is the $n^{th}$ Fibonacci number is a variant of the Riemann zeta function and the analytic continuation of this series was studied by Navas [7]. Unlike $\zeta(s)$, this series has trivial zeros at $-2, -6, -10, \ldots$ and simple poles $0, -4, -8, \ldots$. Also $\zeta_F(s)$ takes rational numbers at negative odd integers. This motivates us to consider the analytic continuation of the series

$$\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s}, \quad \text{Re}(s) > 1$$

(1.1)
where, as usual, \( s = \sigma + it \in \mathbb{C} \) and \( B_n \) is the \( n \)th balancing number [3]. We call the series \( \zeta_B(s) \) the balancing zeta function. We will show that \( \zeta_B(s) \) has simple poles at \( 0, -2, -4, \ldots \) and can be extended to a meromorphic function on \( \mathbb{C} \). However, the balancing zeta function has no trivial zeros unlike the Riemann zeta function.

2. ANALYTIC CONTINUATION OF BALANCING ZETA FUNCTION

Analytic continuation of a series consists of extending the domain of analyticity of the given series. Clearly, the balancing zeta function is analytic in the half plane \( \text{Re}(s) > 1 \). In this section, we will show that the balancing zeta function can be extended to the whole complex plane except at poles.

It is known that the balancing numbers satisfy the recurrence relation

\[
B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 1 \text{ with } B_0 = 0, B_1 = 1
\]

and the Binet form is given by

\[
B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}
\]

where \( \lambda_1 = 3 + 2\sqrt{2} \) and \( \lambda_2 = \frac{1}{\lambda_1} \). For any complex number \( z \)

\[
B_z = \left( \frac{\lambda_1 - \lambda_2}{4\sqrt{2}} \right)^z = 2^{-5z/2} \left( \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \left( 1 - \left( \frac{\lambda_2}{\lambda_1} \right)^n \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \left( 1 - \left( \frac{1}{\lambda_2^n} \right) \right)^z
\]

\[
= 2^{-5z/2} \lambda_1^nz \sum_{k=0}^{\infty} \binom{z}{k} \lambda_1^{-2nk}
\]

\[
= 2^{-5z/2} \sum_{k=0}^{\infty} \binom{z}{k} \lambda_1^{n(z-2k)}.
\]

This expression is valid for any \( z \in \mathbb{C} \) and this binomial series converges since \( \lambda_1 > 1 \). Substituting the final expression for \( B_z \) with \( z = -s \) in (1.1) we get,

\[
\sum_{n=1}^{\infty} B_n^{-s} = 2^{5s/2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} \lambda_1^{n(-s-2k)}.
\]

(2.1)

Thus,

\[
\sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| \binom{-s}{k} \lambda_1^{n(-s-2k)} \right| = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| \lambda_1^{n(-s-2k)} \right|
\]

\[
\leq \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{|s|}{k} \lambda_1^{-n(\sigma+2k)}
\]