A CENTRAL LIMIT THEOREM FOR A NEW STATISTIC ON PERMUATIONS

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This paper does three things: It proves a central limit theorem for novel permutation statistics (for example, the number of descents plus the number of descents in the inverse). It provides a clear illustration of a new approach to proving central limit theorems more generally. It gives us an opportunity to acknowledge the work of our teacher and friend B. V. Rao.

Key words: Metrics on permutations; descents; Stein’s method.

1. INTRODUCTION

Let \( S_n \) be the group of all \( n! \) permutations of \( \{1, \ldots, n\} \). A variety of statistics \( T(\pi) \) are used to enable tasks such as tests of randomness of a time series, comparison of voter profiles when candidates are ranked, non-parametric statistical tests and evaluation of search engine rankings. A basic feature of a permutation is a local ‘up-down’ pattern. Let the number of descents be defined as

\[
D(\pi) := |\{i : 1 \leq i \leq n-1, \pi(i+1) < \pi(i)\}|.
\]

For example, when \( n = 10 \), the permutation \( \pi = (7 1 5 3 10 8 6 2 4 9) \) has \( D(\pi) = 5 \).

The enumerative theory of permutations by descents has been intensively studied since Euler. An overview is in Section 3 below. In seeking to make a metric on permutations using descents we were led to study

\[
T(\pi) := D(\pi) + D(\pi^{-1}).
\]  

(1.1)

For a statistician or a probabilist it is natural to ask “Pick \( \pi \in S_n \) uniformly; what is the distribution of \( T(\pi) \)?” While a host of limit theorems are known for \( D(\pi) \), we found \( T(\pi) \) challenging. A main result of this paper establishes a central limit theorem.

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Theorem 1.1 — For $\pi$ chosen uniformly from the symmetric group $S_n$, and $T(\pi)$ defined by (1.1), for $n \geq 2$

$$E(T(\pi)) = n - 1, \quad \text{Var}(T(\pi)) = \frac{n + 7}{6} - \frac{1}{n},$$

and, normalized by its mean and variance, $T(\pi)$ has a limiting standard normal distribution.

The proof of Theorem 1.1 uses a method of proving central limit theorems for complicated functions of independent random variables due to Chatterjee [3]. This seems to be a useful extension of Stein’s approach. Indeed, we were unable to prove Theorem 1.1 by standard variations of Stein’s method such as dependency graphs, exchangeable pairs or size-biased couplings. Theorem 1.1 is a special case of the following more general result. Call a statistic $F$ on $S_n$ “local of degree $k$” if $F$ can be expressed as

$$F(\pi) = \sum_{i=0}^{n-k} f_i(\pi),$$

where the quantity $f_i(\pi)$ depends only on the relative ordering of $\pi(i+1), \ldots, \pi(i+k)$. For example, the number of descents is local of degree 2, and the number of peaks is local of degree 3. We will refer to $f_0, \ldots, f_{n-k}$ as the “local components” of $F$.

Theorem 1.2 — Suppose that $F$ and $G$ are local statistics of degree $k$ on $S_n$, as defined above. Suppose further that the absolute values of the local components of $F$ and $G$ are uniformly bounded by 1. Let $\pi$ be be chosen uniformly from $S_n$ and let

$$W := F(\pi) + G(\pi^{-1}).$$

Let $s^2 := \text{Var}(W)$. Then the Wasserstein distance between $(W - E(W))/s$ and the standard normal distribution is bounded by $C(n^{1/2}s^{-2} + ns^{-3})k^3$, where $C$ is a universal constant.

After the first draft of this paper was posted on arXiv, it was brought to our notice that the joint normality of $D(\pi)$, $D(\pi^{-1})$ was proved in Vatutin [27] in 1996 via a technical tour de force with generating functions. The asymptotic normality in Theorem 1.1 follows as a corollary of Vatutin’s theorem. Theorem 1.2 is a new contribution of this paper.

In outline, Section 2 describes metrics on permutations and our motivation for the study of $T(\pi)$. Section 3 reviews the probability and combinatorics of $D(\pi)$ and $T(\pi)$. Section 4 describes Chatterjee’s central limit theorem. Section 5 proves Theorem 1.1. The proof of Theorem 1.2 is in Section 6. Section 7 outlines some other problems where the present approach should work.

2. Metrics on Permutations

A variety of metrics are in widespread use in statistics, machine learning, probability, computer science and the social sciences. They are used in conjunction with statistical tests, evaluation of