Multiplications of maximal rank in the cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract Let $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let $C \subseteq Q$ be a curve of type $(a, b)$ having equation $F = 0$. The main purpose of this paper is to analyze the multiplicative structure of the bi-graded module $H^1_*\mathcal{O}_Q$, in particular to prove that for any $r, s \geq 0$ the multiplication map $H^1\mathcal{O}(r, -s) \xrightarrow{F} H^1\mathcal{O}(r + a, -s + b)$ induced by $F$ has maximal rank for the general $C$ of type $(a, b)$. Interpretations of this problem in the contexts of multilinear algebra and differential algebra are emphasized.

Keywords Line bundles · Bi-graded polynomials · Multiplication and contraction maps · Brill–Noether locus · Tangent cone

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1 Introduction

Tensors enter in many fields of pure and applied mathematics and the most common and useful linear maps on spaces of tensors arise from multiplications and contractions. In this paper we address a quite natural question about multiplication and contraction maps of symmetric tensors. We work over the field of the complex numbers $\mathbb{C}$ and set $V = \mathbb{C}^{m+1}$, $W = \mathbb{C}^{n+1}$. Denote symmetric powers with $S^i$. 
Problem 1.1 Let $\sigma \in S^a V \otimes S^b W$ be some fixed element and $r, t$ be some integers with $r \geq 0$ and $t \geq b$. Consider the linear map

$$S^r V \otimes S^t W^* \overset{\sigma}{\rightarrow} S^{r+a} V \otimes S^{t-b} W^*$$

(1)
defined by multiplication on the first $r$ tensor components and contraction on the last $t$ tensor components. Is this map of maximal rank, for $\sigma$ sufficiently general in $S^a V \otimes S^b W$?

Notice that the question is very easy if $\dim W$ or $\dim V = 1$, indeed in these cases the map (1) is given by multiplication of symmetric tensors or by its dual, the contraction, and it is either injective or surjective. The first non-trivial case of Problem 1.1 appears when $\dim W = \dim V = 2$. The object of this paper is to solve Problem 1.1 in this case. We like also to point out two other equivalent formulations of Problem 1.1. Consider the variables $x = (x_0, \ldots, x_m)$, $y = (y_0, \ldots, y_n)$ and the derivations $\partial = (\partial y_0, \ldots, \partial y_n)$. We denote with $C[x]_i$ the vector space of homogeneous polynomials of degree $i$, for any $i \geq 0$.

Problem 1.2 Consider a differential operator $D \in C[x]_a \otimes C[\partial]_b$ and the linear map

$$D : C[x]_r \otimes C[\partial]_t \rightarrow C[x]_{r+a} \otimes C[\partial]_{t-b}.$$ 

(2)

Is this map of maximal rank if $D$ is sufficiently general in $C[x]_a \otimes C[\partial]_b$?

Now let $\mathbb{P}^m$ and $\mathbb{P}^n$ be projective spaces over $\mathbb{C}$ of dimensions $m$, $n$, respectively, $Q = \mathbb{P}^m \times \mathbb{P}^n$ their product and $\pi_1, \pi_2$ the first and second projection, respectively. Recall that Pic($Q$) $\cong \mathbb{Z} \times \mathbb{Z}$, with basis $\mathcal{O}_Q(1, 0) = \pi_1^* \mathcal{O}_{\mathbb{P}^m}(1)$ and $\mathcal{O}_Q(0, 1) = \pi_2^* \mathcal{O}_{\mathbb{P}^n}(1)$. In these notations, one may consider the following third version of Problem 1.1.

Problem 1.3 Consider the multiplication map

$$H^0 \mathcal{O}_Q(r, -t - n - 1) \overset{\sigma}{\rightarrow} H^0 \mathcal{O}_Q(r + a, -t + b - n - 1), \quad t \geq b$$

(3)

with $\sigma \in H^0 \mathcal{O}_Q(a, b)$ a form of bi-degree $(a, b)$. Is this map of maximal rank if $\sigma$ is sufficiently general in $H^0 \mathcal{O}_Q(a, b)$?

The fact that the three problems above are equivalent is well known. For instance the equivalence of Problem 1.1 and Problem 1.3 is due to the fact that $H^n \mathcal{O}_{\pi_1}(-t - n - 1) = H^n \mathcal{O}_{\pi_2}(-t - n - 1) \cong S^r V \otimes S^t W^*$, for $V = H^0 \mathcal{O}_{\pi_1}(1)$ and $W = H^0 \mathcal{O}_{\pi_2}(1)$, by Künneth formula and Serre duality. Moreover the multiplication $H^n \mathcal{O}_{\pi_1}(-t - n - 1) \overset{\sigma}{\rightarrow} H^n \mathcal{O}_{\pi_2}(t - b - n - 1)$, with $\tau \in H^0 \mathcal{O}_{\pi_2}(b)$, is dual to the multiplication $H^0 \mathcal{O}_{\pi_2}(t - b) \overset{\tau}{\rightarrow} H^0 \mathcal{O}_{\pi_1}(-t - n - 1)$, Denoting by $(y^*)^I$ the dual basis of $y^I = y_0^{i_0} \cdots y_n^{i_n}$, with $i_0 + \cdots + i_n = |I| = t$, this map can be described as follows. For any multi-index $J$ with $|J| = b$, one has $(y^*)^I \cdot y^J = 0$ if $J \not\subset I$ and $(y^*)^I \cdot y^J = (y^*)^{I \setminus J}$ if $J \subset I$. This coincides with the differentiation $D (y^*)^I$, where $D = \partial_{y_0}^{i_0} \cdots \partial_{y_n}^{i_n}$, up to a non zero rational number factor, showing the equivalence of Problems 1.1 and 1.3 with Problem 1.2. In this paper we solve Problem 1.3 for $m = n = 1$, using some deep facts about the geometry of curves on the surface $Q = \mathbb{P}^1 \times \mathbb{P}^1$. We know also how to solve the three problems in case $a = b$ and $n, m$ general, by a very different technique involving the differential operator formulation of Problem 1.2. The full solution of Problems 1.1, 1.2, 1.3 will be the object of future investigations. Notice that the fact that the map (3) has maximal rank in the case of $n = m = 1$ helps understanding the multiplicative structure of the bigraded module $H^1_\mathcal{O}_Q$, and hence that of the Rao module of curves $C \subset Q$ when $Q$ is embedded in $\mathbb{P}^3$, cfr. [1].

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