Generating a staircase starshaped set and its kernel in $\mathbb{R}^3$ from certain staircase convex subsets

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Abstract Let $C = \{C_1, \ldots, C_n\}$ be a family of distinct boxes in $\mathbb{R}^3$ whose intersection graph is a tree, and let $S = C_1 \cup \cdots \cup C_n$. Assume that $S$ is starshaped via staircase paths with corresponding staircase kernel $K$, and let $A$ denote the smallest box containing $K$. Then $K$ is staircase convex and $S \cap A = K$. When $S \neq K$, for every point $p$ in $S \setminus K$ define $W_p = \{s : s \text{ lies on some staircase path in } S \text{ from } p \text{ to a point of } K\}$. Each set $W_p$ is staircase convex. Further, there is a minimal collection $\mathcal{W}$ of $W_p$ sets whose union is $S$. The collection $\mathcal{W}$ is unique and finite, and $\mathcal{W}$ is exactly the collection of maximal $W_p$ sets in $S$. Finally, $\bigcap\{W : W \in \mathcal{W}\}$ is exactly the kernel $K$.

Keywords Orthogonal polytope · Staircase convex set · Staircase starshaped set

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1 Introduction

We begin with some definitions from Breen (2010, 2011a,b). A nonempty set $C$ in $\mathbb{R}^d$ is called a box if and only if $C$ is a convex polytope (possibly degenerate) whose edges are parallel to the coordinate axes. A set $S$ in $\mathbb{R}^d$ is an orthogonal polytope if and only if $S$ is a connected union of finitely many boxes. Let $\lambda$ be a simple polygonal path in $\mathbb{R}^d$ whose edges are parallel to the coordinate axes. For $s, t$ in $S$, the path $\lambda$ is called an $s - t$ path in $S$ if and only if $\lambda$ lies in $S$ and has endpoints $s$ and $t$. In this case, $\lambda(s, t)$ will represent the path $\lambda$, ordered from $s$ to $t$. The path $\lambda$ is an $s - t$ geodesic in $S$ if and only if $\lambda$ is an $s - t$ path of minimal length in $S$. (Clearly, an $s - t$
geodesic need not be unique.) The path \( \lambda \) is a \textit{staircase path} (or simply a staircase) if and only if no two of its edges have opposite directions. That is, for each standard basis vector \( e_i \), \( 1 \leq i \leq d \), all edges of \( \lambda \) parallel to \( e_i \) have the same direction. For convenience of notation, we use \( e_i \) or \(-e_i\) to indicate the associated direction. Clearly, if \( \lambda \) is a staircase path in \( S \) with endpoints \( s \) and \( t \), then \( \lambda \) is an \( s-t \) geodesic in \( S \). Moreover, if \( S \) contains an \( s-t \) staircase path, then every \( s-t \) geodesic in \( S \) is an \( s-t \) staircase.

For points \( s \) and \( t \) in \( \mathbb{R}^d \), we define their \textit{distance} \( \text{dist} \,(s,t) \) to be the length of an \( s-t \) geodesic in \( \mathbb{R}^d \). (Of course, every \( s-t \) geodesic in \( \mathbb{R}^d \) is a staircase.) As usual, for \( p \in \mathbb{R}^d \) and \( A \subseteq \mathbb{R}^d \), \( \text{dist} \,(p,A) = \inf \{\text{dist} \,(p,a) : a \in A\} \). For \( p \) a point in set \( S \subseteq \mathbb{R}^d \) and \( A \subseteq \mathbb{R}^d \), we say that the \textit{distance to} \( A \) \textit{is locally maximal at} \( p \) if and only if there is some neighborhood \( N \) of \( p \) such that \( \text{dist} \,(t,A) \leq \text{dist} \,(p,A) \) for every \( t \) in \( N \cap S \).

For points \( s \) and \( t \) in a set \( S \), we say \( s \) \textit{sees} \( t \) (\( s \) is visible from \( t \)) via \textit{staircase paths} if and only if there is a staircase path in \( S \) that contains both \( s \) and \( t \). A set \( S \) is \textit{staircase convex} (orthogonally convex) if and only if, for every pair \( s, t \) in \( S \), \( s \) sees \( t \) via staircase paths. Similarly, a set \( S \) is \textit{staircase starshaped} (orthogonally starshaped) if and only if, for some point \( p \) in \( S \), \( p \) sees each point of \( S \) via staircase paths. The set of all such points \( p \) is the \textit{staircase kernel} of \( S \).

We will use a few standard terms from graph theory. For \( F = \{C_1, \ldots, C_n\} \) a finite collection of distinct sets, the \textit{intersection graph} \( G \) of \( F \) has vertex set \( c_1, \ldots, c_n \). Further, for \( 1 \leq i \leq j \leq n \), the points \( c_i, c_j \) determine an edge in \( G \) if and only if the corresponding sets \( C_i, C_j \) in \( F \) have a nonempty intersection. A graph \( G \) is a \textit{tree} if and only if \( G \) is connected and acyclic. A sequence \( v_1, \ldots, v_k \) of vertices in \( G \) is a \textit{walk} if and only if each consecutive pair \( v_i, v_{i+1} \) determines an edge of \( G \), \( 1 \leq i \leq k-1 \). A walk \( v_1, \ldots, v_n \) is a \textit{path} if and only if its points are distinct.

Finally, for \( C_1, \ldots, C_k \) a collection of distinct boxes in \( \mathbb{R}^d \), we say that their union is a \textit{chain} of boxes (relative to our ordering) if and only if the intersection graph of \( \{C_1, \ldots, C_k\} \) is the path \( c_1, \ldots, c_k \) (where \( c_i \) represents the set \( C_i \) in the intersection graph, \( 1 \leq i \leq k \)). That is, relative to our labeling, for \( 1 \leq i < j \leq k \), \( C_i \cap C_j \neq \emptyset \) if and only if \( j = i+1 \).

Many results in convexity that involve the usual notion of visibility via straight line segments have interesting analogues that instead use the idea of visibility via staircase paths. For example, the familiar Krasnosel’skii theorem Krasnosel’skii (1946) says that, for a nonempty compact set \( S \) in the plane, \( S \) is starshaped via segments if and only if every three points of \( S \) see via segments in \( S \) a common point. In the staircase analogue Breen (1994), for a nonempty simply connected orthogonal polygon \( S \) in \( \mathbb{R}^2 \), \( S \) is staircase starshaped if and only if every two points of \( S \) see via staircase paths in \( S \) a common point. Furthermore, in an interesting study involving rectilinear spaces, Chepoi (1996) has generalized the planar result to any finite union \( S \) of boxes in \( \mathbb{R}^d \) whose intersection graph is a tree. In Breen (2010), some other planar results are extended to such a set \( S \) as well.

In Martini and Wenzel (2003, 2004), results give a starshaped version of the Krein-Milman theorem, presenting a minimal subset of the boundary of a compact starshaped set that, in a sense, generates the whole set. (See also Martini and Wenzel (2005).) Another starshaped version of the Krein-Milman theorem is derived in

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