Maximality on fuzzy filters of lattices

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Received: 26 May 2010 / Accepted: 7 December 2010 / Published online: 11 February 2011
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Abstract In this work, we consider a mimetic definition of maximality for fuzzy filters of lattices, and look for some characterization of this definition. We find out that maximality can be handled by very few elementary properties of fuzzy filters.

Keywords Fuzzy set · Lattice · Fuzzy maximal filter · Maximal fuzzy filter · Congruence

Mathematics Subject Classification (2000) 06B10 · 06D99

1 Introduction

The notion of fuzzy subset was introduced by L.A. Zadeh [8] in the sixties: a fuzzy subset of a set $E$ is a map $f : E \to I$, where $I := [0; 1]$ is the closed interval of real numbers. Since then, a lot of work has been done on fuzzy mathematical structures; most of the authors use the above original definition of a fuzzy set.

In the present work, we replace the closed interval by an arbitrary bounded distributive lattice. So, a fuzzy subset of $E$ will be a map $g : E \to M$, where $M$ is the underlying set of a bounded distributive lattice.

The notions of fuzzy filter and fuzzy ideal have been studied by many authors, especially for ordered structures and rings (cf. [2–5,7]). In this work we study two aspects of maximality on fuzzy filters of lattices. In Sect. 1, we recall some basic facts about fuzzy filters, and fix some notations. Section 2 deals with fuzzy maximal filters, where we obtain a very nice characterization. Section 3 is concerned with maximal fuzzy filters. It appears that for a large class of lattices, maximal fuzzy strictly implies fuzzy maximal. And, if the codomain
of the fuzzy set $\mu$ is a dense lattice, then there is no maximal fuzzy filter. Section 4 gives some observations on (fuzzy) congruences of lattices. Basic notions on lattices, filters and congruences can be found in [1].

2 Preliminaries

Let $\mathcal{L} = (L; \sqcap, \sqcup, \bot, \top)$ and $\mathcal{M} = (M; \land, \lor, 0, 1)$ be two bounded lattices, with corresponding orders $\leq$ and $\leq$ respectively. We also assume that $\mathcal{M}$ is complete.

A nonempty subset $F$ of $L$ is called a filter of $\mathcal{L}$ if for all $x, y \in L$, $(x \sqcap y \in F)$ and $(x \in F$ implies $x \sqcup y \in F)$. For any subset $X$ of $L$, let $(X)$ denote the filter of $\mathcal{L}$ generated by $X$.

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For any $\alpha \in M$, let $\mathcal{c}_\alpha$ denote the constant map $L \to M$ with value $\alpha$. Let $M^L$ denote the set of maps from $L$ to $M$, and define the order relation $\leq$ on $M^L$ pointwise; i.e., $f \leq g$ iff $\forall x \in L$, $f(x) \leq g(x)$. Then $(M^L; \leq)$ is a lattice, which is complete since $\mathcal{M}$ is, where for any family $\{f_i; i \in I\}$ in $M^L$, $(\bigvee_{i \in I} f_i)(x) := \bigvee_{i \in I} f_i(x)$.

**Definition 2.1** (i) A map $\mu : L \to M$, is called an $\mathcal{M}$-fuzzy subset of $L$.

(ii) A fuzzy subset $\mu$ is called proper if it is not a constant map.

(iii) If $\mu : L \to M$ is a fuzzy subset of $L$ and $\alpha \in M$, then $\mu_\alpha := \{x \in L; \mu(x) \geq \alpha\}$ is called the $\alpha$-cut set of $\mu$.

If $h : L \to M$ is a fuzzy subset of $L$, and $\alpha, \beta$ are elements of $\text{Im}(h)$; then $h_\alpha = h_\beta$ implies that $\alpha = \beta$.

**Definition 2.2** (i) A fuzzy subset $\mu : L \to M$ is called an $\mathcal{M}$-fuzzy sublattice of $\mathcal{L}$ if for each $\alpha \in M$, $\mu_\alpha \neq \emptyset$ implies $\mu_\alpha$ is a sublattice of $\mathcal{L}$.

(ii) A fuzzy subset $\mu$ is called an $\mathcal{M}$-fuzzy filter of $\mathcal{L}$ if for each $\alpha \in M$, $\mu_\alpha \neq \emptyset$ implies $\mu_\alpha$ is a filter of $\mathcal{L}$.

It is well known that [3,4] a fuzzy subset $\mu : L \to M$ is

(i) a fuzzy sublattice iff $\forall x, y \in L$, $\mu(x \sqcup y) \land \mu(x \sqcap y) \geq \mu(x) \land \mu(y)$.

(ii) a fuzzy filter iff $\forall x, y \in L$, $\mu(x) \leq \mu(x \sqcup y)$ and $\mu(x \sqcap y) \geq \mu(x) \land \mu(y)$.

**Example 2.1** Let $E$ be a set with at least three elements $a$, $b$ and $c$, and $L := \mathcal{P}(E)$ be the power set of $E$. Consider the lattice $\mathcal{L} := (L; \cap, \cup, \emptyset, E)$, and the three maps $f$, $g$, $h$ from $L$ to $L$ defined as follows for each $X \in L$:

$f(X) = X \cup \{a\}$.

$g(X) = \begin{cases} 
\{a\} & \text{if } a \in X \\
\{b\} & \text{if } a \notin X \text{ and } b \in X \\
\{c\} & \text{if } \{a, b\} \cap X = \emptyset.
\end{cases}$

$h(X) = \begin{cases} 
X \cup \{a\} & \text{if } c \in X \\
X & \text{if } c \notin X.
\end{cases}$