Additive Schwarz method for the $p$-version of the boundary element method for the single layer potential operator on a plane screen

Norbert Heuer
Universidad de Concepción, Departamento de Ingeniería Matemática, Casilla 160-C, Concepción, Chile; e-mail: norbert@ing-mat.udec.cl

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Dedicated to Prof. Dr. Ernst P. Stephan on the occasion of his 50th birthday.

Summary. We analyze an additive Schwarz preconditioner for the $p$-version of the boundary element method for the single layer potential operator on a plane screen in the three-dimensional Euclidean space. We decompose the ansatz space, which consists of piecewise polynomials of degree $p$ on a mesh of size $h$, by introducing a coarse mesh of size $H \geq h$. After subtraction of the coarse subspace of piecewise constant functions on the coarse mesh this results in local subspaces of piecewise polynomials living only on elements of size $H$. This decomposition yields a preconditioner which bounds the spectral condition number of the stiffness matrix by $O(\log \frac{H}{h} p)^2$. Numerical results supporting the theory are presented.

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1 Introduction

In this paper we analyze a preconditioner of the additive Schwarz type for linear systems arising from the Galerkin method for symmetric, positive definite pseudo-differential operators of order minus one on surfaces in the three-dimensional Euclidean space. As a model problem we consider the single layer potential operator $V$ on a plane screen $\Gamma \subset \mathbb{R}^3$:

For given $g \in H^{1/2}(\Gamma)$ find $u \in \tilde{H}^{-1/2}(\Gamma)$ such that $Vu = g$, i.e.

\begin{equation}
\tag{1} a(u, v) := \langle Vu, v \rangle_{L^2(\Gamma)} = \langle g, v \rangle_{L^2(\Gamma)} \quad \text{for all } v \in \tilde{H}^{-1/2}(\Gamma)
\end{equation}
where

\[ \text{Vu}(x) := \frac{1}{4\pi} \int_{\Gamma} \frac{u(y)}{|x-y|} \, dy, \quad x \in \Gamma. \]

For the definition of the Sobolev spaces \(H^{1/2}(\Gamma)\) and \(H^{-1/2}(\Gamma)\) see Sect. 3. \(V\) is a continuous and symmetric, positive definite mapping from \(H^{-1/2}(\Gamma)\) to \(H^{1/2}(\Gamma)\), cf. \([13, 37]\), and therefore \(V\) defines a norm which is equivalent to the \(H^{-1/2}\)-norm.

Physically the solution \(u\) of (1) represents the charge density of a thin, electrified plate \(\Gamma\) loaded with some given potential. We note that the results in this paper hold for polyhedral surfaces \(\Gamma\) as well.

The Galerkin scheme for (1) reads: For a given finite-dimensional subspace \(X_N \subset H^{-1/2}(\Gamma)\) find \(u_N \in X_N\) such that for every \(v \in X_N\)

\[ a(u_N, v) = \langle g, v \rangle_{L^2(\Gamma)}. \]  

Here we focus on the \(p\)-version of the boundary element method which reduces the approximation error \(\|u - u_N\|_V\) by increasing the polynomial degrees of the ansatz functions on a given mesh \(\Gamma_h\). We use uniform meshes which consist of rectangles of length \(h\). Since the conformity condition \(X_N \subset H^{-1/2}(\Gamma)\) does not require continuous functions we simply use affine images of tensor products of \(L^2\)-normed Legendre polynomials as basis functions, i.e. on the reference element \((-1, 1)^2\) we take the functions

\[ \sqrt{(2q_1 + 1)(2q_2 + 1)} L_{q_1}(x_1) L_{q_2}(x_2), \quad \max\{q_1, q_2\} \leq p. \]

We note that only for ease of presentation and in order to give full details of the proofs we deal with rectangular meshes and basis functions of the tensor product type. The main ingredients in the proof of the main theorem are estimates of different Sobolev norms for general functions on Lipschitz domains, scaling properties of Sobolev norms, and the inverse property of basis functions. All the needed details we expect to hold for more general, quasi-uniform and regular, quadrilateral and triangular meshes and other basis functions as well.

It is well known that the solution \(u\) of (1) behaves singularly at the edges and corners of \(\Gamma\), cf. \([39]\). Therefore the \(p\)-version converges twice as fast as the \(h\)-version, where one uses piecewise polynomials of low degree on a sequence of refined meshes, cf. \([38]\). On the other hand the spectral condition number \(\kappa\) of the stiffness matrix \(A\) in (2) behaves like \(O(h^{-1}p^2)\) and, therefore, increases much faster for the \(p\)-version than for the \(h\)-version. This behavior of the condition number occurs when taking basis functions that are orthonormal with respect to the \(L^2\)-inner product. A proof of this fact uses the characterization of the extreme eigenvalues by the Rayleigh quotient and the inverse property of the basis functions (see Lemma 4 below). For