HYBRID FEM WITH FUNDAMENTAL SOLUTIONS AS TRIAL FUNCTIONS FOR HEAT CONDUCTION SIMULATION

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ABSTRACT A new type of hybrid finite element formulation with fundamental solutions as internal interpolation functions, named as HFS-FEM, is presented in this paper and used for solving two dimensional heat conduction problems in single and multi-layer materials. In the proposed approach, a new variational functional is firstly constructed for the proposed HFS-FE model and the related existence of extremum is presented. Then, the assumed internal potential field constructed by the linear combination of fundamental solutions at points outside the elemental domain under consideration is used as the internal interpolation function, which analytically satisfies the governing equation within each element. As a result, the domain integrals in the variational functional formulation can be converted into the boundary integrals which can significantly simplify the calculation of the element stiffness matrix. The independent frame field is also introduced to guarantee the inter-element continuity and the stationary condition of the new variational functional is used to obtain the final stiffness equations. The proposed method inherits the advantages of the hybrid Trefftz finite element method (HT-FEM) over the conventional finite element method (FEM) and boundary element method (BEM), and avoids the difficulty in selecting appropriate terms of T-complete functions used in HT-FEM, as the fundamental solutions contain usually one term only, rather than a series containing infinitely many terms. Further, the fundamental solutions of a problem are, in general, easier to derive than the T-complete functions of that problem. Finally, several examples are presented to assess the performance of the proposed method, and the obtained numerical results show good numerical accuracy and remarkable insensitivity to mesh distortion.

KEY WORDS hybrid FEM, fundamental solution, variational functional, heat conduction

I. INTRODUCTION

During the past decades, research into the development of efficient finite elements has been mostly concentrated on the following three distinct types of FEM\([1-6]\). The first is the conventional FEM. It is based on a suitable polynomial interpolation which has already been used to analyze most engineering problems. With this method, the solution domain is divided into a number of small cells or elements, and material properties are defined at element level\([1]\). The second is the natural-mode FEM. In contrast, the natural FEM, initiated by Argyris et al.\([2]\), presents a significant alternative to conventional FEM with ramifications on all aspects of structural analysis. It makes distinction between the constitutive and geometric parts of the element tangent stiffness, which could lead effortlessly to the non-linear

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effects associated with large displacements. When applied to composite structures, the physically clear and comprehensible theory with complete quadrature elimination and avoidance of modal (shape) functions can show distinctly the mechanical behavior of isotropic and composite shell structures\cite{2,3}. The final is the so-called hybrid Trefftz FEM (HT-FEM)\cite{4,6}. Unlike in the conventional and natural FEM, the HT-FEM couples the advantages of FEM\cite{1} and BEM\cite{7}. In contrast to conventional FEM and BEM, HT-FEM is based on a hybrid method which includes the use of an independent auxiliary inter-element frame field defined on each element boundary and an independent internal field chosen so as to a priori satisfy the homogeneous governing differential equations by means of a suitable truncated T-complete function set of homogeneous solutions. Inter-element continuity is enforced by using a modified variational principle, which is used to construct the standard force-displacement relationship, that is, stiffness equation, and establish linkage of frame filed and internal fields of the element. The property of nonsingular element boundary integral that appears in HT-FEM enables us to construct arbitrarily shaped element conveniently; however, the terms of truncated T-complete functions should be carefully selected in achieving desired results. Further, they are difficult to develop for some physical problems. To remove the drawback of HT-FEM, a novel hybrid finite formulation based on the fundamental solution, named as HFS-FEM, is firstly developed for solving two dimensional heat conduction problems in single and multilayer-materials. The proposed HFS-FEM can be viewed as the fourth type of FEM which is significantly different from the previous three types of FEM discussed above. In the analysis, a linear combination of the fundamental solution at different points is used to approximate the field variable within the element. The independent frame field defined along the element boundary and the newly developed variational functional are employed to guarantee the inter-element continuity, generate the final stiffness equation and establish linkage between the boundary frame field and internal field in the element. The proposed HFS-FEM inherits all advantages of HT-FEM and removes the difficulty in constructing and selecting T-functions, so it can reach more extensive applications than the HT-FEM. Moreover, it is necessary to point out that the developed approach is different from the BEM, although the same fundamental solution is employed. Using the reciprocal theorem, the BEM obtains the boundary integral equation, which usually encounters difficulty in dealing with singular or hyper-singular integrals in the BEM analysis, while the proposed method can remove this weakness. Additionally, the more flexible element material definition in the HFS-FEM is important for multi-material analysis, rather than the material definition in the entire domain in the BEM.

The paper begins with a simple description of heat conduction problems in §II. Then, a detailed derivation of the proposed HFS-FEM and the corresponding algorithm is described in §III to provide an initial insight on this new finite element model. Several numerical examples are presented in §IV to validate the proposed algorithm and some concluding remarks are presented in §V.

II. STATEMENT OF HEAT CONDUCTION PROBLEMS

Consider that we are seeking to find the solution of a well-posed heat conduction problem in a general plane domain $\Omega$

\[
\frac{\partial}{\partial X_1} \left( k \frac{\partial u(x)}{\partial X_1} \right) + \frac{\partial}{\partial X_2} \left( k \frac{\partial u(x)}{\partial X_2} \right) = 0 \quad \forall x \in \Omega \tag{1}
\]

with the following boundary conditions:

—Dirichlet boundary condition related to the unknown temperature field

\[
u = \bar{u} \quad \text{on } \Gamma_u
\tag{2}

—Neumann boundary condition for the boundary heat flux

\[
q = -k u_i n_i = \bar{q} \quad \text{on } \Gamma_q
\tag{3}
\]

where $k$ stands for the thermal conductivity, $u$ is the sought field variable and $q$ represents the boundary heat flux. $n_i$ is the $i$th component of the outward normal vector to the boundary $\Gamma = \Gamma_u \cup \Gamma_q$, and $\bar{u}$ and $\bar{q}$ are specified functions on the related boundaries, respectively. The space derivatives are indicated by a comma, i.e. $u_i = \partial u/\partial X_i$, and the subscript index $i$ takes values 1 and 2 in our analysis. Additionally, the repeated subscript indices stand for the summation convention.