Linearity properties of Shimura varieties, II

BEN MOONEN
Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstraße 62, 48149
Münster, Germany

Abstract. Let $A$ denote the moduli scheme over $\mathbb{Z}[1/n]$ of p.p. $g$-dimensional abelian varieties
with a level $n$ structure; its generic fibre can be described as a Shimura variety. We study its ‘Shimura
subvarieties’. If $x \in A$ is an ordinary moduli point in characteristic $p$, then we formulate a local
‘linearity property’ in terms of the Serre–Tate group structure on the formal deformation space ($=\$-
formal completion of $A$ at $x$). We prove that an irreducible algebraic subvariety of $A$ is a ‘Shimura
subvariety’ if, locally at an ordinary point $x$, it is ‘formally linear’. We show that there is a close
connection to a differential-geometrical linearity property in characteristic 0.

We apply our results to the study of Oort’s conjecture on subvarieties $Z \hookrightarrow A$ with a dense
collection of CM-points. We give a reformulation of this conjecture, and we prove it in a special case.


Key words: Shimura varieties, Serre–Tate theory, Oort’s conjecture.

Introduction

In this second part of our work on ‘linearity properties’, which for a large part is
independent of Part I (= [19]), we continue our investigation of subvarieties of
Hodge type in a given Shimura variety. We study their properties, and in particular
the question of how such subvarieties of Hodge type can be characterized. Here we
are not looking for a description of all subvarieties (which can be given in terms of
the Deligne formalism of Shimura varieties) but rather for direct characterizations
of when an algebraic subvariety $Z \hookrightarrow \text{Sh}_K(G, x)$ is of Hodge type. The conjectures
of Coleman and Oort (see the introduction of Part I; for Oort’s conjecture see also
below and Section 5) can be seen as motivating problems.

In this paper, we restrict our attention to subvarieties $Z \hookrightarrow A_{g,1,n}$ of the moduli
space of $g$-dimensional abelian varieties (+ polarization and a level structure).
Similar to Th. 4.3 in Part I, we prove that an algebraic subvariety $Z$ is of Hodge
type if and only if it satisfies a certain ‘linearity property’. In this case, however,
we work with a linearity property (called ‘formal linearity’) of more arithmetic
flavour than the ‘total geodesicness’ considered in Part I. The set-up is as follows.

Let $Z \hookrightarrow A_{g,1,n} \otimes F$ be an absolutely irreducible algebraic subvariety of $A_{g,1,n}$
over a number field $F$. Let $p$ be a prime of $\mathcal{O}_F$ not dividing $n$, write $A_g = A_{g,1,n} \otimes \mathcal{O}_p$, and define $Z \hookrightarrow A_g$ as the Zariski closure of $Z$ inside $A_g$. Suppose
$x \in Z \otimes \kappa(p)$ is a closed ordinary moduli point. Taking formal completions
at the point $x$, we obtain formal schemes $\mathfrak{Z}_x \hookrightarrow \mathfrak{A}_x$ over $\text{Spf}(\Lambda)$, where $\Lambda =$
$W(\kappa(x)) \otimes_{W(\kappa(p))} \hat{\mathcal{O}}_p$. By Serre–Tate theory, $\mathfrak{a}_x$ has a natural structure of a formal torus. We say that $Z$ is formally linear at $x$ if $\mathfrak{a}_x \hookrightarrow \mathfrak{a}_x$ is a formal subtorus.

Our study of this notion of formal linearity was motivated by results of Rutger Noot ([21], see also [22]). He proved that if we start with a subvariety $Z$ of Hodge type, then the model $Z$ as above is formally linear at all closed ordinary points $x$ (possibly excluding finitely many primes $p$). We give a precise formulation of Noot’s results in Section 4.

One of our objects in this paper is to prove that, conversely, a weakened version of formal linearity implies that the subvariety $Z$ is of Hodge type. More precisely:

**THEOREM.** Let $Z \hookrightarrow \mathbb{A}_{g,1,n} \otimes F$ be an irreducible algebraic subvariety of the moduli space $\mathbb{A}_{g,1,n}$, defined over a number field $F$. Suppose there is a prime $p$ of $\mathcal{O}_F$ such that the model $Z$ of $Z$ (as in Section 3.3) has formally quasi-linear components at some closed ordinary point $x \in (Z \otimes \kappa(p))^\circ$. Then $Z$ is of Hodge type, i.e., every irreducible component of $Z \otimes_F \mathbb{C}$ is a subvariety of Hodge type.

Notice that this statement is very similar to Corollary 5.5 in Part I, to which, in fact, we reduce the proof. The definitions and preliminaries that are needed to establish the main results, are discussed in Sections 1–3. The above theorem is proved in Section 4.

Next we try to apply our characterization to Oort’s conjecture. Recall that this conjecture says that an irreducible algebraic subvariety $Z \hookrightarrow \mathbb{A}_{g,1,n} \otimes \mathbb{C}$ containing a Zariski dense collection of CM-points should be a subvariety of Hodge type. What we would like to prove therefore, is that the existence of such a Zariski dense collection of CM-points implies that, for some prime $p$, we obtain a model $Z$ which is formally linear at some ordinary point $x$. Unfortunately, we can only prove this under an additional assumption. Although this does not settle Oort’s conjecture in general, we think that both the methods used and the resulting variant of the conjecture (see 5.3) are interesting in their own right.

We conclude with some applications. In Section 5 we apply the main results discussed above to prove Oort’s conjecture in a particular situation. For a precise statement, see 5.7. In the last section we study the Zariski closure of the moduli point of $X^{\text{can}}$, where $X$ is an ordinary abelian variety in characteristic $p$ (not necessarily defined over a finite field). First we show that this Zariski closure, call it $Z$, is a subvariety of Hodge type. Knowing this, one wonders how $\dim(Z)$ compares to the dimension of the Zariski closure $\{x\}^{\text{Zar}} \subset \mathbb{A}_{g,1,n} \otimes F_p$ of the moduli point of $X$. Clearly, if $x$ is a closed point, then both dimensions are zero. In general, $\dim(Z) \geq \dim(\{x\}^{\text{Zar}})$. We show (joint work with A.J. de Jong and F. Oort) that there exist ordinary moduli points $x$ with $\dim(\{x\}^{\text{Zar}}) = 1$ and $\dim(Z) = g(g+1)/2$. 
