On the monodromy at infinity of a polynomial map, II

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Abstract. To a polynomial map $f: \mathbb{C}^{n+1} \to \mathbb{C}$ one can attach a monodromy transformation on the complex cohomology of its generic fiber that reflects its asymptotic behaviour. In this paper this transformation is determined for a class of generic polynomials in terms of data attached to projective hypersurfaces with isolated singularities.

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1. Introduction

In the last years a lot of work has been concentrated on the study of the behaviour at infinity of polynomial maps (see for example [27], [28], [3], [4], [29], [13], [15]–[17], [30], [22], [31], [9], among others). This behaviour can be very complicated, therefore one of the main ideas was to find special classes of polynomial maps which have, in some sense, nice properties at infinity. In this paper, we completely determine the complex algebraic monodromy at infinity for a special class of polynomial maps (which is complicated enough to show the nature of the general problem).

Next, we give the precise definitions: Let $f: \mathbb{C}^{n+1} \to \mathbb{C}$ be a map given by a polynomial with complex coefficients (which will be also denoted by $f$). Then there exists a finite set $\Gamma \subset \mathbb{C}$ such that the map

$f|_{\mathbb{C}^{n+1} - f^{-1}(\Gamma)}: \mathbb{C}^{n+1} - f^{-1}(\Gamma) \to \mathbb{C} - \Gamma$

is a locally trivial $C^\infty$-fibration ([18]). We denote by $\Gamma_f$ the smallest subset of the complex plane with this property. $\Gamma_f$ contains the set $\Sigma_f$ of critical values of $f$, but in general it is bigger. Fix $t_0 \in \mathbb{C}$ such that $|t_0| > \max\{|t|: t \in \Gamma_f\}$. The complex

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algebraic monodromy associated with the path $s \mapsto t_0 e^{2\pi i s}$, $s \in [0, 1]$, is denoted by

$$(T_f^\infty)^*: H^s(\text{f}^{-1}(t_0), \mathbb{C}) \to H^s(\text{f}^{-1}(t_0), \mathbb{C}).$$

This isomorphism is called the monodromy at infinity of $f$. As we will see later, $(T_f^\infty)^*$ is a very delicate invariant of $f$.

On studying topological properties of polynomial maps, one usually imposes some condition which insures the absence of vanishing cycles ‘at infinity’ for a suitable compactification of the map $f$ (tameness, Malgrange condition, . . ., cf. e.g. [3] and [30]). From this point of view, a class of polynomial maps which looks natural to study is the following.

DEFINITION. A polynomial $f \in \mathbb{C}[X_1, \ldots, X_{n+1}]$ will be called a $(*)$-polynomial if it verifies the following condition

$$(*) \quad \begin{cases} 
\text{For } t \in \mathbb{C} - \Sigma_f, \text{ the closure in } \mathbb{P}^{n+1} \text{ of the affine} \\
\text{hypersurface } \{f = t\} \text{ is non-singular.}
\end{cases}
$$

The goal of this article is the computation of $(T_f^\infty)^*$ for $(*)$-polynomials.

We will assume that $n \geq 2$. The case $n = 1$ is completely clarified in [9].

If $d = \text{deg}(f)$ and $f = f_d + f_{d-1} + \cdots$ is the decomposition of $f$ into homogeneous components, condition $(*)$ is equivalent to

$$\{x \in \mathbb{C}^{n+1} \mid \text{grad} f_d(x) = 0, \ f_{d-1}(x) = 0\} = \{0\},$$

where $\text{grad}$ denotes the gradient vector. The local analogue of $(*)$-polynomials are the superisolated singularities, introduced by Luengo in [14]. The local algebraic monodromy of superisolated surface singularities was determined by E. Artal in [1].

In the first part [9] of this sequence of papers, the following results are given (besides others):

(a) A $(*)$-polynomial $f$ satisfies $\Gamma_f = \Sigma_f$ and any fiber of $f$ has the homotopy type of a bouquet of $n$-dimensional spheres (cf. [7]). In particular, the only interesting monodromy transformation is $(T_f^\infty)^n$, which in the sequel will be denoted simply by $T_f^\infty$.

(b) The hypersurface $X^\infty \subset \mathbb{P}^n$ given by $f_d = 0$ has only isolated singularities, and the monodromy at infinity (actually, the whole topology at infinity) depends only on the hypersurface $X^\infty$. Thus the study of the monodromy at infinity of $(*)$-polynomials is very strongly linked with the study of projective hypersurfaces with isolated singularities. This link gives results in both directions (see below the Main Theorem, Corollary 3 and the comments after it).