Mazur’s incidence structure for projective varieties (I)

BIN WANG
Mathematics Department, Yale University, New Haven, CT 06520–8283, U.S.A.
e-mail: binwang@minerva.cis.yale.edu

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Abstract. Let $X$ be an $m$ dimensional smooth projective variety with a Kähler metric. We construct a metrized line bundle $L$ with a rational section $s$ over the product $\mathcal{C}_p(X) \times \mathcal{C}_q(X)$ of Chow varieties $\mathcal{C}_p(X), \mathcal{C}_q(X)$ such that

$$\frac{1}{(m-1)!} \log |s(A, B)|^2 = \langle A, B \rangle$$

for disjoint $A, B$. That gives an answer to a part of Barry Mazur’s proposal in a private communication to Bruno Harris about the Archimedean height pairing $\langle A, B \rangle$ on a smooth projective variety $X$.


Key words: Archimedean height pairing, Green’s current, Chow variety.

1. Introduction

In higher dimensional arithmetic geometry, Archimedean height pairing originated from Arakelov’s work ([A1], [A2]) was first introduced by Bloch ([B]), Beilinson ([Be]). It is the local pairing at infinity. It was studied by Gillet, Soulé, Bost, and Hain in various contexts. On the other hand, the Chow variety has a longer history that goes back to Cayley, Chow, and Chow coordinates have always been useful in Diophantine geometry ([BGS], [P1], [P2], [P3]). But as for the geometry of the Chow variety, we seem to be so far from understanding it. In 1993, aiming at the geometry of Chow varieties Barry Mazur proposed series of questions ([M]) concerning the incidence relation induced from the Archimedean height pairing.

To view it in a naive way, for a smooth projective variety $X$ of dimension $m$, we take the incidence set $\mathcal{D} = \{(A, B) \in \mathcal{C}_p(X) \times \mathcal{C}_q(X) : A \cap B \neq \emptyset\}$, where $\mathcal{C}_p(X)$ and $\mathcal{C}_q(X)$ are the Chow varieties of dimension $p$ and $q$ with $p + q = m - 1$. The question is: is $\mathcal{D}$ a variety, a divisor, or even more, is it a Cartier divisor? Mazur’s proposal is to relate the $\mathcal{D}$ to the Archimedean height pairing. This is based on the fact that Archimedean height pairing is a continuous function outside of $\mathcal{D}$ in $\mathcal{C}_p(X) \times \mathcal{C}_q(X)$, but as the pair of disjoint cycles $A \times B$ approaches...
the Archimedean height pairing of $A$, $B$ goes to infinity ([W3] or [W4]). In the special case where $X$ is a projective space, and $p = 0$, $q = m - 1$, $D$ is the classical universal hypersurface. In general, to obtain a divisor, we need to associate a ‘multiplicity’ to each $(A, B) \in C_p(X) \times C_q(X)$, which shall reflect the way how $A$ and $B$ intersect. In [M], a Weil divisor $D_w$ supported on the incidence set $D$ is defined by using intersection theory. Two of the questions asked by Mazur are:

(1) Is $D_w$ a Cartier divisor?
(2) If it is, let $\mathcal{L}$ be the corresponding line bundle and $s$ the section that defines $D_w$. Does there exist a metric $\| \cdot \|$ on $\mathcal{L}$, such that

$$\log \|s\|^2(A, B) = \langle A, B \rangle$$

for any disjoint cycles $A \in C_p(X)$, $B \in C_q(X)$, where $\langle A, B \rangle$ is the Archimedean height pairing?

1.1. THE STATEMENT

The main purpose of this paper is to construct a line bundle and a section $s$ through the calculation of the Archimedean height pairing, which give a formula close to (1.0) (see 1.1.3 below).

DEFINITION 1.1.1. Let $X$ be a smooth, irreducible projective variety of dimension $m$ with a Kähler metric. Let $A, B$ be a linking pair of effective cycles in $X$ of dimensions $p$ and $q$ respectively, i.e. $p + q = m - 1$, and $A$, $B$ are disjoint. If $A$ and $B$ are disjoint irreducible subvarieties in $X$, the Archimedean height pairing $\langle A, B \rangle$ of $A$ and $B$ is defined to be the integral $\int_A [G_B]$ (it is well-defined because of the condition (2) of the smoothness below) of a Green’s current of $B$ over $A$. A Green’s current ([Bo], 1.2) of $B$ is a current $[G_B]$ of type $(p, p)$ on $X$ satisfying:

(1) As currents $(i/2\pi) \partial \bar{\partial}[G_B] = \delta_B - [\omega_B]$, where $\delta_B$ is the current of integration over $B$ and $[\omega_B]$ is the current of a smooth form $\omega_B$ on $X$;
(2) $[G_B]$ is smooth outside of $B$;
(3) (Normalization) $\omega_B$ is harmonic and the harmonic projection of $[G_B]$ is zero.

If, furthermore, $[G_B]$ can be represented by an $L^1$ form that is smooth on $X \setminus B$, the $L^1$ form will be denoted by $G_B$ called ‘Green’s form’.

We linearly extend the definition of Archimedean height pairing, and Green’s currents to cycles. Let $C^\alpha_r(X)$ denote the Chow variety of $X$ which parametrizes the effective cycles of dimension $r$ in $X$ with cohomology class $\alpha$. By linear extension one can view Archimedean Height Pairing as a function on an open subset $\mathcal{U}$ of the product $C^\alpha_p(X) \times C^\alpha_q(X)$ of the Chow varieties, where $\mathcal{U}$ consists of all the disjoint pairs of cycles.