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Topos Perspective on the Kochen–Specker Theorem: III. Von Neumann Algebras as the Base Category

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We extend the topos-theoretic treatment given in previous papers of assigning values to quantities in quantum theory, and of related issues such as the Kochen–Specker theorem. This extension has two main parts: the use of von Neumann algebras as a base category and the relation of our generalized valuations to (i) the assignment to quantities of intervals of real numbers and (ii) the idea of a subobject of the coarse-graining presheaf.

1. INTRODUCTION

Two previous papers [1, 2] have developed a topos-theoretic perspective on the assignment of values to quantities in quantum theory. In particular, it was shown that the Kochen–Specker theorem (which states the impossibility of assigning to each bounded self-adjoint operator on a Hilbert space of dimension greater than 2 a real number such that functional relations are preserved) is equivalent to the nonexistence of a global element of a certain presheaf $\mathbf{\Sigma}$, called the spectral presheaf, defined on the category $\mathcal{O}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$. In particular, the Kochen–Specker theorem's *FUNC* condition—which states that assigned values preserve the operators' functional relations—turns out to be equivalent to the 'matching condition' in the definition of a global section of the spectral presheaf. It was similarly shown that the Kochen–Specker theorem is equivalent to the

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nonexistence of a global element of a presheaf **D**—called the dual presheaf—defined on the category $\mathcal{W}$ of Boolean subalgebras of the lattice $\mathcal{L}(\mathcal{H})$ of projectors on $\mathcal{H}$.

It was also shown that it *was* possible to define so-called *generalized valuations* on all quantities according to which any proposition "$A \in \Delta$" (read as saying that the value of A lies in the Borel set of real numbers Δ) is assigned, in effect, a set of quantities that are coarse-grainings (functions) of A. To be precise, it is assigned a certain set of morphisms in the category $\mathcal{O}$ (or $\mathcal{W}$), the set being required to have the structure of a *sieve*. These generalized valuations obey a condition analogous to *FUNC* and other natural conditions. Furthermore, each (pure or mixed) quantum state defines such a valuation.

In this paper, we will extend this treatment in two main ways. The first corresponds to our previous concerns with the Kochen–Specker theorem and global sections and with generalized valuations based on sieves. Thus, we will first discuss the issues adumbrated above in terms of a base category different from $\mathcal{O}$ and $\mathcal{W}$: namely, the category $\mathcal{V}$ of commutative von Neumann subalgebras of an algebra of operators (Sections 2 and 3).

Second, we will further develop the idea of a generalized valuation (Section 4). In particular, we introduce the idea of an interval-valued valuation: at its simplest, the idea is to assign to a quantity A—not an individual member of its spectrum, as vetoed by the Kochen–Specker theorem—but rather, some subset of it. Though this idea seems at first sight very different from generalized valuations that assign sieves to propositions "$A \in \Delta$," we shall see that the two types of valuations turn out to be closely related.

2. VON NEUMANN ALGEBRAS

2.1. Introducing $\mathcal{V}$

We will first rehearse the definitions given in the previous papers [1, 2] of the categories $\mathcal{O}$ and $\mathcal{W}$ defined in terms of the operators on a Hilbert space over which various presheaves may be usefully constructed. Then we will introduce a new base category $\mathcal{V}$ which has as objects commutative von Neumann algebras and relate it to $\mathcal{O}$ and $\mathcal{W}$.

The categories $\mathcal{O}$ and $\mathcal{W}$ were defined as follows. The objects of the category $\mathcal{O}$ are the bounded self-adjoint operators on the Hilbert space $\mathcal{H}$ of some quantum system. A morphism $f_{\mathcal{O}}: \hat{B} \to \hat{A}$ is defined to exist if $\hat{B} = f(\hat{A})$ [in the sense of ref. 1, Eq. (2.4)] for some Borel function f. This category is a preorder, and may be turned into a partially ordered set by forming equivalence classes of operators. Operators $\hat{A}$ and $\hat{B}$ are considered equivalent whenever they are isomorphic in the category $\mathcal{O}$, i.e., when there exist some