ON RATES AND LIMIT DISTRIBUTIONS

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Abstract. For regular parametric models, estimators converge uniformly at
a rate $n^{-1/2}$, and the limit distribution is normal with mean 0. The situation
is different if the best possible rate is $n^{-\alpha}$, with $\alpha \in (0, 1/2)$, as common
for nonparametric models. In this case, uniformly attainable normal limit
distributions with mean 0 are impossible.

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1. Introduction

Let $(X, A)$ be a measurable space, and $\mathfrak{P}$ a family of probability measures
$P \mid A$. The problem is to estimate a functional $\kappa : \mathfrak{P} \to \mathbb{R}$, based on a sample
of size $n$, distributed according to $P^n$, with $P \in \mathfrak{P}$ unknown. For $n \in \mathbb{N}$ let
$\hat{\kappa}^{(n)} : X^n \to \mathbb{R}$ be an estimator.

For parametric models we are used to the existence of estimator sequences
with the following properties. (i) They converge with the rate $n^{1/2}$, uniformly on compact subsets of the parameter space, to a normal distribution, (ii) this normal distribution is maximally concentrated among all limit distributions which can be attained locally uniformly.

The situation is not so favourable in the nonparametric case. Checking the
literature, one finds rate bounds, say $c_n, n \in \mathbb{N}$, for various models. That means:
If for some estimator sequence $\hat{\kappa}^{(n)}, n \in \mathbb{N}$, the sequence of standardized errors,
$\hat{r}_n(\kappa^{(n)} - \kappa(P))$, is under $I^n$ asymptotically bounded, uniformly on $\mathfrak{P}$, then $c_n,
 n \in \mathbb{N}$, cannot converge to infinity quicker than $c_n, n \in \mathbb{N}$. In other words: $c_n,
 n \in \mathbb{N}$, is the best possible rate for the convergence of $\kappa^{(n)}$ to $\kappa(P)$. A rate
bound is not necessarily attainable. To show that $c_n, n \in \mathbb{N}$, is, in fact, a possible
rate, it would be necessary to find an estimator sequence, say $\kappa^{(n)}_0, n \in \mathbb{N}$, such
that $\hat{r}_n(\kappa^{(n)}_0 - \kappa(P)), n \in \mathbb{N}$, is under $I^n$ asymptotically bounded, uniformly on $\mathfrak{P}$. More often than not, the authors are satisfied with something quite different,
namely: The existence of an estimator sequence converging with the rate $c_n, n \in \mathbb{N}$,
to a limit distribution pointwise. That means: The distribution of \( c_n(\kappa_{\alpha}^{(n)} - \kappa(P)) \) under \( P^n \) converges to a limit distribution for every \( P \in \mathfrak{I} \).

Since (local) uniformity is a constitutive element in the definition of a rate bound, convergence to a limit distribution does not establish that this rate is attainable, unless the convergence to the limit distribution is (locally) uniform.

The reader should keep in mind that there may be several estimators attaining the rate bound \( c_n, n \in \mathbb{N} \), differing by an amount which is stochastically of the order \( O(c_n) \). Hence limit distributions attained with the rate \( c_n, n \in \mathbb{N} \), —if any—are not unique.

The careful distinction between the convergence rate of \( \kappa^{(n)} \) to \( \kappa \), and the rate at which the distribution of \( \kappa^{(n)} - \kappa(P) \) under \( P^n \) converges to a limit distribution is without much use in case of regular parametric models. Here, the distribution of \( \kappa^{(n)} - \kappa(P) \) converges with the rate \( n^{1/2} \) uniformly to a limit, and this implies that the rate \( n^{1/2} \) is attainable. In nonparametric models, the uniform rate bounds for the convergence of \( \kappa^{(n)} \) to \( \kappa(P) \) are usually of type \( c_n - n^{1/2} \) with \( c \in (0, 1/2) \). Even if there is an estimator sequence converging to \( \kappa \) at this rate, this does not imply convergence at this rate to a limit distribution, let alone uniform convergence.

Rate bounds for the convergence to \( \kappa \) can be obtained by different methods. In the present paper, we make no assumptions about the origin of the rate bounds. Within this restricted framework, one can show that limit distributions attained with a convergence rate \( n^{1/2} \) with \( \alpha \in (0, 1/2) \) and \( L \), a slowly varying function, cannot be uniformly attained if they have expectation zero and a finite absolute moment of order \( (1 - \alpha)^{-1} \). Under more specific assumptions on the nature of the rate bounds it will be shown elsewhere (see Pfanzagl (1998)) that uniformly attainable limit distributions and confidence intervals with uniform covering probabilities are impossible if \( \alpha \in (0, 1/2) \).

As a side result, we obtain that limit distributions are necessarily normal if they are (i) locally uniformly attained with a rate \( n^{1/2} \) and (ii) maximally concentrated on arbitrary intervals containing 0.

Our method of proof is not suitable to deal with the case \( \alpha > 1/2 \). It is, however, clear that comparable results are not to be expected. For the purpose of illustration we mention the family of uniform distributions over \( (0, \theta) \), say \( P_{\theta} \). The sequence \( \theta - \max \{ x_1, \ldots, x_n \} \) converges with the rate \( n \) to a non-normal limit distribution, namely the exponential distribution. According to Millar ([1983], pp. 156–157) this distribution is maximally concentrated among all uniformly attainable limit distributions.

Theorem 2.1 in Section 2 is the main result. Applications to the case \( \alpha = 1/2 \) and \( \alpha \in (0, 1/2) \) are given in Sections 3 and 4, respectively. Section 5 contains a detailed discussion of the concept of a rate bound. Various technical lemmas and the proof of Theorem 2.1 are given in Section 6.

2. The main result

We shall use the following notation. For any probability measure \( P | A \) and a measurable function \( h : X \rightarrow \mathbb{R} \), \( P \circ h \) denotes the induced distribution on \( \mathbb{R} \), defined by \( P \circ h(B) := P(h^{-1}(B)), B \subset \mathbb{R} \).