Continuous Accumulation Games in Continuous Regions¹,²

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Abstract. In a continuous accumulation game on a continuous region, a Hider distributes material over a continuous region at each instant of discrete time, and a Seeker examines the region. If the Seeker locates any of the material hidden, the Seeker confiscates it. The goal of the Hider is to accumulate a certain amount of material before a given time, and the goal of the Seeker is to prevent this. In previous works, we have studied accumulation games involving discrete objects and continuous material over discrete locations. The issues raised when the region is continuous are substantially different. In this paper, we study accumulation of continuous material over two types of continuous regions: the interval and the circle.

Key Words. Search games, two-person zero-sum games, accumulation games.

1. Introduction

In a continuous accumulation game on a continuous region (CAGCR), a team known as Hider distributes material over a continuous region at each instant of discrete time (turn), and a team called Seeker examines the region. The goal of the Hider is to accumulate a certain amount of material before a given time, and the goal of the Seeker is to prevent this by finding and confiscating material. In a previous work (Ref. 1), we have studied accumulation games involving discrete objects hidden in discrete locations.

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We refer the reader to that paper for a sketch of the history of such problems and a bibliography of related works.

The issues raised when the region is continuous are substantially different from the discrete case due to the topological properties of continuous regions.

Example 1.1. The Hider hides $h$ units of material on the circumference $C$ of a circle having length one in such a way that the upper boundary forms a continuous nonnegative function $f$ on $C$, with

$$\int_C f(t) \, dt = h.$$ 

The Seeker may search once a connected arc having length $s < 1$ and win, with payoff 1 if it finds any part of the Hider material. This game has an $\varepsilon$-value equal to $s$. If the Seeker chooses the starting point for the search arc according to a random distribution on the circle, then with probability at least $s$ this search arc will contain a point $u$ on the circle with $f(u) > 0$. If the search arc contains such a point $u$, then the Seeker will find a positive amount of the Hider material, because $f$ is continuous. Thus, the Seeker will win with probability at least $s$. On the other hand, if the Hider chooses a point $x$ on the circumference at random and then concentrates its material over an arc of length $t$ beginning at $x$, for instance by using a function with graph an isosceles triangle of height $2h/t$, then this arc will intersect the arc of the Seeker with probability at most $t/Cs$. As $t$ converges to 0, this quantity converges to $s$. Thus, the Hider can hold the expected payoff to the Seeker as close to $s$ as the Hider desires.

Example 1.2. We shall modify slightly the model of Example 1.1 and obtain a totally different result. The Hider can use the same strategies, but the Seeker may use any open set. By a well-known exercise in real analysis, there is a dense open set $S$ of $C$ such that the measure of $S$ is equal to $s$ no matter how small $s$ is. Suppose that the Seeker chooses the set $S$. If the maximum of the function $f$ on $C$ is $M$, then $S$ has a nonempty open intersection with the set

$$D = \{x : f(x) > M/2\}.$$ 

Then, we shall have

$$\int_{S \cap D} f(x) \, dx > (M/2)m(S \cap D) > 0,$$