Multipermutation-Based Intersection Theorem and Its Applications

Z. F. Yang

Communicated by F. Zirilli

Abstract. In this paper, we introduce a multipermutation-based intersection theorem on the product space of several unit simplices, called the simplotope. This theorem gives a substantial generalization of an intersection result of Scarf on the unit simplex (Ref. 1). By using this new result, we also obtain a multipermutation-based generalization of the Brouwer fixed-point theorem on the simplotope. Furthermore, we apply this new result to an economic equilibrium model with indivisibilities and obtain an equilibrium existence theorem.

Key Words. Intersection theorem, fixed-point theorem, combinatorial theorem, equilibrium problem, indivisibility, equilibrium theorem.

1. Introduction

Intersection theorems state conditions under which a collection of sets has an intersection point. These theorems are related closely to the Brouwer fixed-point theorem and Sperner lemma, and are often used to prove the existence of solutions to economic equilibrium problems and game theoretic problems; see for example Scarf (Ref. 1), Shapley (Ref. 2), Gule (Ref. 3), Talman (Ref. 4), van der Laan, Talman, and Yang (Ref. 5), Herings and Talman (Ref. 6). The most well-known intersection lemma is probably that of Knaster, Kuratowski, and Mazurkiewicz (Ref. 7), KKM lemma in short.

Let $S^n$ be the $(n-1)$-dimensional unit simplex, being a subset of the nonnegative orthant $\mathbb{R}^n_+$ of the $n$-dimensional Euclidean space, where the sum of the components equals one. The KKM lemma says that, if $S^n$ is
covered by a collection of closed sets \( C_1, C_2, \ldots, C_n \) such that, when for \( x \) in \( S^n \) there exists an index \( i \in \{1, 2, \ldots, n\} \) with \( x_i > 0 \) and \( x \in C_i \), then these \( n \) sets have a nonempty intersection. The same conclusion holds when, for every point \( x \) in \( S^n \), \( x_i = 0 \) implies \( x \in C_i \). The latter result is due to Scarf (Ref. 1). We can derive easily the Brouwer fixed-point theorem by using the Scarf intersection lemma.

Let \( f \) be a continuous function from \( S^n \) into itself. For each \( i \), define the set
\[
C_i = \{x \in S^n \mid f_i(x) \geq x_i\}.
\]
It is readily seen that all the conditions of the Scarf lemma are satisfied. Hence, there exist a point \( x^* \in S^n \) such that
\[
f_i(x^*) \geq x^*_i, \quad \text{for all } i.
\]
Obviously,
\[
f(x^*) = x^*.
\]

In this paper, we will introduce a new intersection theorem on the product space of several unit simplices \( S^n \), called the simploptope. This theorem will be called a multipermutation-based intersection theorem, which leads to a substantial generalization of the Scarf intersection lemma. Moreover, a multipermutation-based generalization of the Brouwer fixed-point theorem will be derived from this new intersection theorem. Furthermore, we apply this new result to an economic equilibrium model with indivisibilities and obtain an equilibrium existence theorem.

The paper is organized as follows. In Section 2, we present our main results. In Section 3, we discuss an application of these results in the context of economic equilibrium theory.

**2. Main Results**

First, we introduce some notation. Let \( k, m, n \) be positive integers. Let \( I_k \) denote the set of the first \( k \) positive integers, and let \( I_k^n \) denote \( I_k \times \cdots \times I_k \), where \( I_k \) is repeated \( k \) times. Define
\[
\Phi = \{\rho \mid \rho = (\rho(1), \ldots, \rho(n)) \text{ is a permutation of } (1, \ldots, n)\}.
\]
An element \( \pi \in \prod_{j=1}^n \Phi \) is written as \( \pi = (\pi_1, \ldots, \pi_m) \), where \( \pi_j = (\pi_j(1), \ldots, \pi_j(n)) \) for each \( j \in I_m \). For any \( x, y \in \mathbb{R}^n \), \( \langle x, y \rangle \) denotes the inner product of \( x \) and \( y \). The vector \( e(l) \) is the \( l \)th unit vector of \( \mathbb{R}^n \) for each \( l \in I_n \). The vector \( e \) denotes a vector in \( \mathbb{R}^n \) whose components are all