AN ELEMENTARY PROOF OF PÓLYA–VINOGRAVODOV'S INEQUALITY, II

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Abstract

Let $\chi$ be a primitive character mod $k$, $k > 2$. In [1], the following elementary estimate

$$s(\chi) \leq \begin{cases} \frac{1}{\pi} \sqrt{k} \log k + \left(1 - \frac{\log 2}{\pi}\right) \sqrt{k} + \frac{1}{2}, & \text{if } \chi(-1) = 1, \\ \frac{1}{\pi} \sqrt{k} \log k + \sqrt{k} + \frac{1}{2}, & \text{if } \chi(-1) = -1, \end{cases}$$

was given, where

$$s(\chi) = \max_{r \geq 1} \left| \sum_{n=1}^{r} \chi(n) \right|$$

by definition. In the present note we sharpen this estimate by a factor $3/4$ in the case of an even primitive character $\chi$, by improving upon the proof given in [1] in a way which does not alter the elementary character of the method.

We shall prove the following

**Theorem 1.** Let $\chi$ be an even primitive character mod $k$, $k > 2$. Then

$$s(\chi) \leq \frac{3}{4\pi} \sqrt{k} \log k + \left(2 - \frac{\log 2}{\pi} - \frac{\gamma}{2\pi}\right) \sqrt{k} + 1,$$

where $\gamma = 0.577\ldots$ is Euler's constant.

**Proof.** Start with the following identity proved in [1]

$$G(1, \chi) \sum_{n=1}^{r} \chi(n) = \sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k}}{\sin \frac{\pi m}{k}} - \sum_{m \leq \frac{k}{2}} \chi(m)$$

$$+ \sum_{m \leq \frac{k}{2}} \chi(m) \cos \frac{2\pi rm}{k}.$$  

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We need the following lemma.

**Lemma 1.** Let \( \chi \) be an even primitive character mod \( k \), \( k > 2 \), and let \( t \) be a non-negative integer. Then

\[
\sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} = \sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos^{1+2t} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} + \Delta_t,
\]

where, \( |\Delta_t| \leq t\sqrt{k} \).

**Proof.** We prove the lemma by induction on \( t \). The lemma is trivially true for \( t = 0 \). Assume that it is true for \( t > 0 \). It is easily seen that

\[
\frac{\sin \frac{2\pi rm}{k} \cos^{1+2t} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} = \frac{\sin \frac{2\pi rm}{k} \cos^{1+2(t+1)} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} - \cos \frac{2\pi rm}{k} \cos^{2+2t} \frac{\pi m}{k} \frac{\pi (2r - 1)m}{k} \cos^{1+2t} \frac{\pi m}{k},
\]

in consequence

\[
\sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos^{1+2t} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} = \sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos^{1+2(t+1)} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} - \sum_{m \leq \frac{k}{2}} \chi(m) \cos \frac{\pi (2r - 1)m}{k} \cos^{1+2t} \frac{\pi m}{k}.$

Induction hypothesis implies then

\[
(2) \quad \sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} - \sum_{m \leq \frac{k}{2}} \chi(m) \frac{\sin \frac{2\pi rm}{k} \cos^{1+2(t+1)} \frac{\pi m}{k}}{\sin \frac{\pi m}{k}} =
\]

\[
- \sum_{m \leq \frac{k}{2}} \chi(m) \cos \frac{2\pi rm}{k} \cos^{2+2t} \frac{\pi m}{k} + \sum_{m \leq \frac{k}{2}} \chi(m) \cos \frac{\pi (2r - 1)m}{k} \cos^{1+2t} \frac{\pi m}{k} + \Delta_t.
\]

The successive application of the identity

\[2 \cos x \cos y = \cos(x + y) + \cos(x - y)\]

shows that both the sums in the right member of (2) are linear combinations of Gaussian sums; both combinations are \( \leq \sqrt{k}/2 \) in absolute value. It follows that the absolute value of the right member of (2) is not greater than

\[
\frac{\sqrt{k}}{2} + \frac{\sqrt{k}}{2} + |\Delta_t| \leq \frac{\sqrt{k}}{2} + \frac{\sqrt{k}}{2} + t\sqrt{k} = (t + 1)\sqrt{k}.
\]