HYPERPLANE TRANSVERSAALS OF HOMOTHETICAL, CENTRALLY SYMMETRIC POLYTOPES

HORST MARTINI (Chernitz) and ANITA SCHÖBEL (Kaiserslautern)

Abstract

Let $P \subseteq \mathbb{R}^n$ be a centrally symmetric, convex $n$-polytope with $2r$ vertices, $n \geq 2$. Let $\mathcal{P}$ be a family of $m \geq n + 1$ homothetical copies of $P$. Based on an algorithmical approach to center hyperplanes of finite point sets in Minkowski spaces with polyhedral norms, we show that a hyperplane transversal of all members of $\mathcal{P}$ (if it exists) can be found in $O(rm)$ time when the dimension $n$ is fixed.

1. Introduction

Let $\mathcal{C} := \{C_1, \ldots, C_m\}$ be a family of convex sets in $n$-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 2$. Then a hyperplane $H \subseteq \mathbb{R}^n$ is said to be a hyperplane transversal (or a stabbing hyperplane) with respect to $\mathcal{C}$ if $H \cap C_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$. The hyperplane transversal problem is to find out whether there exist hyperplane transversals with respect to such a family $\mathcal{C}$. (Note that the replacement of the hyperplane above by a $k$-dimensional affine flat, $1 \leq k \leq n - 1$, yields the more general $k$-flat transversal problem.)

In this paper we solve a special case of the hyperplane transversal problem, namely the case when $\mathcal{C}$ is restricted to a family $\mathcal{P}$ of $m \geq n + 1$ scaled translates of a centrally symmetric, convex $n$-polytope $P$ with $2r$ vertices (i.e., $\mathcal{P}$ consists of $m$ homothetical copies of $P$). To exclude trivial subcases, we will always assume that the affine hull of the centers of all polytopes from $\mathcal{P}$ is $n$-dimensional. We will show that this problem can be solved in $O(rm)$ time for any fixed dimension $n \geq 2$, by using a result from location theory. In particular, we show that the hyperplane transversal problem discussed here can be modelled as a hyperplane location problem with center objective function (see section 3), which is an extension of the point set width problem.

Mathematics subject classification numbers, 52B11, 52B12, 68Q25.
Key words and phrases. Center hyperplane, centrally symmetric polytope, common transversal, hyperplane transversal, Minkowski space, polyhedral norm, scaled translates, stabbing problem.
With respect to the $k$-flat transversal problem, the following results are known. In the planar case a line transversal of a family $\mathcal{C}$ of $m$ convex sets can be found in $O(m \log m)$ time (cf. [12]), and this time complexity has been shown to be optimal by [8], even in the case when all members of $\mathcal{C}$ are translates of each other. If, in addition, the members of $\mathcal{C}$ are pairwise disjoint translates of a convex set, then linear time is enough, see [10].

Whereas there are further numerous approaches to line stabbing problems in the plane, only a few algorithms for analogous problems in higher dimensions are known. For example, [1] succeeded in stabbing $m$ line segments in $\mathbb{R}^n$ by a hyperplane with $O(m^n)$ time, and for $n = 3$ a plane stabber for a set of $m$ convex polyhedra with a total of $sm$ vertices can be found in $O(s^2m^3)$ time. For further results on stabbing convex polyhedra in $\mathbb{R}^3$ see [8]. In higher dimensions, e.g. the following results are known: If $\mathcal{P}$ is a family of convex $n$-polytopes having a total of $s$ vertices, then the space of hyperplane transversals of $\mathcal{P}$ can be constructed in $O(s^n)$ time, cf. [9] and [7]. If $\mathcal{P}$ consists of $m$ convex $n$-polytopes with a total of $a$ edges, then a hyperplane transversal of $\mathcal{P}$ can be found in $O(m \cdot a^{n-1})$ time (see [2]), and from [5] and [6] it follows that the same time complexity is sufficient if $a$ denotes the total number of directions determined by polytope edges of all members of $\mathcal{P}$.

More algorithmical approaches to stabbing problems are discussed in section 5 of [11], and mainly theoretical results about $k$-flat transversals (e.g., related to Helly-type theorems) can be found in [21], [4], [26], [27], and [23] (the above mentioned survey [11] contains also a lot of theoretical results). For example, [20] investigates the problem of stabbing boxes in higher dimensions, and related results for general convex polytopes were obtained by [6], [1] and others. However, we could not find our result presented here in the known literature.

As the reader will see below, our transversal problem can be solved with the help of the center hyperplane problem in Minkowski spaces having polyhedral norm. Furthermore, in [17] the strong relation between this location problem and the median hyperplane problem is presented, cf. also [18].

2. Basic notions and a related location problem

Since our result on hyperplane transversals is shown to be strongly related to a result from location science (namely, that of finding center hyperplanes of finite point sets in Minkowski spaces), we have to introduce some notions related to distance measures and finite-dimensional normed spaces. According to [19] (see also [25] for a modern representation) we define norms geometrically, with the help of the respective unit balls. For $x \in \mathbb{R}^n, n \geq 2$, and $B \subset \mathbb{R}^n$ a compact, convex set with nonempty interior and centred at the origin, the norm $\gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\gamma(x) := \min\{\lambda > 0 : x \in \lambda B\}.$$