Consequence Operations
Based on Hypergraph
Satisfiability

Abstract. Four consequence operators based on hypergraph satisfiability are defined. Their properties are explored and interconnections are displayed. Finally their relation to the case of the Classical Propositional Calculus is shown.

Key words: hypergraph, clause, satisfiability, consistency, resolution, compactness, consequence operators.

1. Preliminaries

Let us recall some definitions and facts which can also be found in Cowen [1, 2] and Kolany [3].

A hypergraph is a structure $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set and $\mathcal{E}$ is a family of nonempty subsets of $\mathcal{V}$. The elements of $\mathcal{V}$ will be called vertices, and the elements of the set $\mathcal{E}$, edges of the hypergraph $\mathcal{G}$. Sets of vertices will sometimes be called clauses.

A hypergraph is compact iff every edge contains a finite one.

A hypergraph is locally finite iff every vertex belongs to a finite number of edges only.

A hypergraph is edge disjoint iff its edges are pairwise disjoint.

Notice that a graph is a hypergraph with at most two-element edges.

Here, the vertices of a fixed hypergraph $\mathcal{G}$ will be interpreted as some elementary propositions and the edges of $\mathcal{G}$ will be inconsistent sets of them. This interpretation leads to the following generalization of satisfiability of families of disjunctions.

Definition 1.1. A set of vertices $\sigma$ satisfies a family of clauses $\mathcal{A}$ (wrt. $\mathcal{G}$) iff

1. $\sigma$ does not contain any edge,
2. $\sigma$ meets all clauses of $\mathcal{A}$, that is $\sigma \cap \alpha \neq \emptyset$, for all $\alpha$ in $\mathcal{A}$.
If some \( \sigma \) satisfies the family \( \mathcal{A} \), we say that \( \mathcal{A} \) is \textit{satisfiable} wrt. \( \mathcal{G} \), or \( \mathcal{G} \)-satisfiable, for brevity.

Sets \( \sigma \) which do not contain edges will be called \textbf{consistent}. A set \( \sigma \subseteq \mathcal{V} \) is \textbf{inconsistent} iff it is not consistent.

Sets \( \sigma \) which satisfy all edges will be called \textbf{colorings} of \( \mathcal{G} \).

As it was stated above the notion of the hypergraph satisfiability is a generalization of the satisfiability in the sense of Classical Propositional Calculus. To see this let us consider a hypergraph \( \mathcal{G}_0 = (\mathcal{V}_0, \mathcal{E}_0) \), where \( \mathcal{V}_0 \) is the set of all propositional variables and their negations, and \( \mathcal{E}_0 \) consists of all pairs \( \{p, \neg p\} \), \( p \) is a variable. Moreover, let \( X \) be a family of disjunctions of propositional variables and their negations only. Then the family \( X \) is satisfiable in the usual sense iff \( X \) is \( \mathcal{G}_0 \)-satisfiable, where elements of \( X \) are treated as sets of their disjuncts.

The hypergraph introduced above will be called \textbf{the graph of the Classical Propositional Calculus}, or \textbf{the CPC-graph}, for short.

Hypergraph satisfiability has the following property, analogous to the compactness property of CPC-satisfiability (comp. [3])

**Theorem 1.2.** Let \( \mathcal{G} \) be a compact hypergraph. Then a family of finite clauses \( \mathcal{A} \) is \( \mathcal{G} \)-satisfiable iff every finite subfamily of \( \mathcal{A} \) is \( \mathcal{G} \)-satisfiable.

By the above, the unsatisfiability of a family of finite clauses \( \mathcal{A} \) reduces to the unsatisfiability of some finite subfamily of \( \mathcal{A} \).

There are two syntactical characterizations of the hypergraph satisfiability, both due to R. Cowen – the \textbf{Combinatorial Analytic Tableaux (CAT) method} (cf. [2]) and a resolution like approach (see [1, 3, 4]). Here we recall the latter one.

**Definition 1.3.** Let \( \mathcal{G} \) be a hypergraph, \( \alpha_1, \ldots, \alpha_n \) finite clauses, and let \( e = \{a_1, \ldots, a_n\} \) be an edge of \( \mathcal{G} \). We say that the clause \( \alpha = \bigcup_{j=1}^{n} \alpha_j \setminus \{a_j\} \) results by the \textbf{resolution} on the edge \( e \) iff \( a_j \in \alpha_j \) for \( j = 1, \ldots, n \). If a clause \( \alpha \) results by resolution on an edge \( e \) from clauses \( \alpha_1, \ldots, \alpha_n \), we write \( \alpha_1, \ldots, \alpha_n \vdash_e \alpha \). The closure of a family of clauses \( \mathcal{A} \) under the resolution on all edges of \( \mathcal{G} \) will be denoted as \( [\mathcal{A}]_\mathcal{G} \).

It should be noticed that in the above definition we consider the resolution on finite edges only.

The following analogue of the well known fact was proved in [3]

**Theorem 1.4.** Let \( \mathcal{A} \) be a finite family of finite clauses, then \( \mathcal{A} \) is \( \mathcal{G} \)-satisfiable iff \( \emptyset \notin [\mathcal{A}]_\mathcal{G} \).