Nonosculating Sets of Positive Reach

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Abstract. For integral geometric and local topological properties of two sets of positive reach, one fixed and the other moving or translating, certain exceptional relative positions have to be excluded. We give a measure geometric justification in form of a Sard-type theorem which extends to more general singular sets. A slightly stronger result has been proved by Schneider for the special case of convex bodies.


Key words: sets of positive reach, exceptional relative positions, nonosculating condition.

In a recent paper, R. Schneider [7] called two convex bodies $K$ and $K'$ in $\mathbb{R}^d$ in exceptional relative position if they have a common boundary point at which the linear hulls of their normal cones have a nontrivial intersection. He proved with methods of convex geometry that the set of rigid motions $g$ for which $K$ and $gK'$ are in exceptional relative position is of Haar measure zero.

In [6], we used implicitly a slightly weaker result for the more general class of sets with positive reach. This property is sufficient for the integral geometric applications mentioned in Schneider’s paper. Moreover, under a certain nonosculating condition, a translative version holds true which was used implicitly in [5]. The corresponding result for the motion group is a consequence. In this short note, we will demonstrate a proof by means of the area and co-area formula of Federer.

Let $X$ and $Y$ be two sets of positive reach in $\mathbb{R}^d$ in the sense of Federer [1]. nor $X$ denotes the unit normal bundle of such a set, i.e.

\[ \text{nor } X := \{(x, n) \in \mathbb{R}^d \times S^{d-1} : x \in X, n \in \text{Nor}(X, x)\}, \]

where $\text{Nor}(X, x)$ is the dual convex (normal) cone to the convex (tangent) cone $\text{Tan}(X, x)$ of $X$ at $x$. In [1], Theorem 4.10, the following is shown: If there is no $(x, n) \in \text{nor } X$ such that $(x, -n) \in \text{nor } Y$, then the set $X \cap Y$ has locally positive reach and

\[ \text{Tan}(X \cap Y, x) = \text{Tan}(X, x) \cap \text{Tan}(Y, x) \]

and, consequently,

\[ \text{Nor}(X \cap Y, x) = \text{Nor}(X, x) + \text{Nor}(Y, x) \]
for all \( x \in X \cap Y \). The sets \( X \) and \( Y \) in this case will be called nonosculating.

In [5] we have assumed that

\[
\text{for Lebesgue almost all } z \in \mathbb{R}^d \text{ the sets } X \text{ and } Y + z
\]

(translation by \( z \)) are nonosculating \((\text{NOC})\)

in order to prove a translative integral geometric formula for curvature measures. Note that the sets \( X \) and \( RY \) for almost all rotations \( R \) of \( \mathbb{R}^d \) fulfill the condition \((\text{NOC})\) (see Federer [1], proof of Theorem 6.11). A counterexample for the translation group is constructed in Rataj [4]. The measure geometric translative version of Schneider’s result reads as follows: (\( \mathcal{H}^k \) and \( \mathcal{L}^k \) denote \( k \)-dimensional Hausdorff measure and Lebesgue measure in Euclidean spaces, respectively.)

**THEOREM.** If \( X \) and \( Y \) are sets of positive reach satisfying the nonosculating condition \((\text{NOC})\), then for \( \mathcal{L}^d \)-almost all \( z \in \mathbb{R}^d \) and \( \mathcal{H}^{d-1} \)-almost all \( (x, u) \in \text{nor}(X \cap (Y + z)) \) there are unique \( m \in \text{nor}(X, x) \cap S^{d-1} \), \( n \in \text{nor}(Y + z, x) \cap S^{d-1} \) and \( \alpha \in [0, 1] \) with

\[
u = \frac{am + (1 - \alpha)n}{|am + (1 - \alpha)n|}
\]

and any such representation for \( m, n, \alpha \) as above provides a vector \( u \) from \( \text{nor}(X \cap (Y + z), x) \cap S^{d-1} \).

**Remarks.** (1) The number \( \alpha \) may be considered as a parameter for the vectors \( u \) lying on the shorter arc of the great circle passing from \( m \) to \( n \).

(2) The mapping

\[
(m, n, \alpha) \mapsto \frac{am + (1 - \alpha)n}{|am + (1 - \alpha)n|}
\]

from \( (\text{nor}(X, x) \cap S^{d-1}) \times (\text{nor}(Y, y) \cap S^{d-1}) \times [0, 1] \) into \( S^{d-1} \) is injective iff the intersection of the linear hulls of \( \text{nor}(X, x) \) and \( \text{nor}(Y, y) \) is trivial.

(3) The kinematic intersection formula for the generalized curvature measures proved in Glasauer [3], Theorem 3.1, for two convex bodies (which stimulated the paper [7]) may be derived from Theorem 1 in [5] for sets of positive reach: Choosing there \( h(z, x, u) = 1_{f(A \times B \times [0, 1])}(x, x - z, u) \) for Borel sets \( A, B \) in \( \mathbb{R}^d \times S^{d-1} \) (wich has a measurable version) one obtains the translative version. Integration over the rotations of \( Y \) yields the kinematic formula.

**Proof.** Recall that for all \( z \) as in the assumption \((\text{NOC})\) and all \( x \in X \cap (Y + z) \) we have

\[
\text{nor}(X \cap (Y + z), x) = \text{nor}(X, x) + \text{nor}(Y + z, x)
\]