A Classification of 2-Dimensional Laguerre Planes
Admitting 3-Dimensional Groups of
Automorphisms in the Kernel

Dedicated to H. Salzmann on the occasion of his 70th birthday

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Abstract. We determine, up to isomorphisms, all 2-dimensional Laguerre planes that admit
3-dimensional groups of automorphisms in the kernel of the action on parallel classes.

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1. Introduction

A Laguerre plane \( \mathcal{L} = (P, C, ||) \) is an incidence structure consisting of a point set \( P \), a
circle set \( C \) and an equivalence relation \( || \) (parallelism) defined on the point set such
that three mutually non-parallel points can be joined by a unique circle; such that
the circles which touch a fixed circle \( K \) at \( p \in K \) partition \( P \setminus \{p\} \), where \( \{p\} \) denotes
the parallel class of \( p \); such that each parallel class meets each circle in a unique
point (parallel projection); and such that there is a circle that contains at least three
points (richness); compare [1] and [2].

In this paper we are only concerned with Laguerre planes whose common point set
is the cylinder \( Z = S^1 \times \mathbb{R} \) (where \( S^1 \) usually is represented as \( \mathbb{R} \cup \{\infty\} \)), whose
circles are graphs of functions \( S^1 \to \mathbb{R} \) and whose parallel classes of points are
the verticals on the cylinder. Notice that for an incidence structure on the cylinder
with circles and parallel classes like this the axioms of parallel projection and
richness are automatically satisfied. In particular, we are interested in 2-dimensional
or flat Laguerre planes on the cylinder. These Laguerre planes are characterised by
the fact that all their circles are graphs of continuous functions from \( S^1 \) to \( \mathbb{R} \);
cf. [1], [2].
The classical 2-dimensional Laguerre plane is obtained as the geometry of non-trivial plane sections of a cylinder in \( \mathbb{R}^3 \) with an ellipse in \( \mathbb{R}^2 \) as base, or equivalently, as the geometry of non-trivial plane sections of an elliptic cone, in real projective three-space, with its vertex removed. The parallel classes are the generators of the cylinder or cone. By replacing the ellipse in this construction by arbitrary ovals in \( \mathbb{R}^2 \), that is, convex, differentiable simply closed curves, we also obtain 2-dimensional Laguerre-planes. These are the so-called 2-dimensional ovoidal Laguerre planes.

Circles of 2-dimensional Laguerre planes, as described above, are homeomorphic to the unit circle \( S^1 \). When the circle sets are topologized by the Hausdorff metric with respect to a metric that induces the topology of the point set, then the planes are topological in the sense that the operations of joining three points by a circle, intersecting of two circles, and touching are continuous with respect to the induced topologies on their respective domains of definition. For more information on topological Laguerre planes we refer to [1] and [2].

An automorphism of a Laguerre plane is a permutation of the point set such that parallel classes are mapped to parallel classes and circles are mapped to circles. Every automorphism of a 2-dimensional Laguerre plane is continuous and thus a homeomorphism of \( \mathbb{Z} \). The collection of all automorphisms of a 2-dimensional Laguerre plane \( L \) forms a group with respect to composition, the automorphism group \( \Gamma \) of \( L \). This group is a Lie group of dimension at most 7 with respect to the compact-open topology, see [11]. The kernel \( T \) of a Laguerre plane consists of all automorphisms that fix each parallel class, that is, \( T \) is the kernel of the action of \( \Gamma \) on the set of parallel classes. This collection of automorphisms is a closed normal subgroup of the automorphism group of the Laguerre plane. The dimension of the kernel of a 2-dimensional Laguerre plane is at most 4 and the planes with 4-dimensional kernel are precisely the ovoidal Laguerre planes; cf. Theorem 1 below. In this paper we determine all 2-dimensional Laguerre planes that admit a kernel of dimension at least 3. We show that the family of Laguerre planes \( L(f; g, h, k) \) constructed in [9] comprises, up to isomorphisms, all 2-dimensional Laguerre planes whose kernels are at least 3-dimensional. More precisely, these planes are obtained for \( h = \text{id} \), \( k = 1 \), or \( f = g \), \( k = 1 \), or \( f = g \), \( h = \text{id} \); see the following section for a description of the Laguerre planes \( L(f; g, h, k) \).

2. Preliminaries and Main Theorem

With every point \( p \) of a 2-dimensional Laguerre plane \( L \) we can associate a derived incidence structure, called the derived affine plane \( A_p = (A_p, L_p) \) at \( p \), whose point set \( A_p \cong \mathbb{R}^2 \) consists of all points of \( L \) that are not parallel to \( p \) and whose line set \( L_p \) consists of all restrictions to \( A_p \) of circles of \( L \) passing through \( p \) and of all parallel classes not passing through \( p \). Indeed, each derived affine plane \( A_p \) of a 2-dimensional Laguerre plane is even a topological affine plane and extends to a 2-dimensional compact projective plane \( P_p \), which we call the derived projective