COMPACTIFICATION OF $L(Q)$

ABSTRACT. In this paper we extend the usual notion of model (as a structure) to the more general notion of Cauchy Sequence of Structures in a similar way as rationals are extending to real numbers by means of Cauchy sequences of rationals. We show that the structure space $St^\tau$ is dense in the complete space $CSt^\tau$ of Cauchy sequences of structures and that $CSt^\tau$ is compact in the (topo)logical sense.

1. INTRODUCTION

In this paper $L^\tau(Q)$ denotes the first order logic of finite relational type $\tau$ with the quantifier (monadic for simplicity) $Q$ and with usual semantics, that is, if $A$ is a structure with domain $A$ and $q(A)$ is the interpretation of the quantifier $q$ in the structure $A$, then, $A \models Qx\varphi(x)$ means \{a $\in A$ : $A \models \varphi[a]\} \in q(A)$. $St^\tau$ denotes the class of structures of type $\tau$ and it is a proper class. In what follows, we will study the topological structure of the space $St^\tau$. If $A, B \in St^\tau$, we say that $A$ and $B$ are $n$-equivalents, in symbols, $A \equiv_n B$, if $A \models \varphi$ iff $B \models \varphi$ for all sentence $\varphi \in L^\tau(Q)$ with quantificational rank $qr(\varphi) \leq n$, where $qr(\varphi)$ is defined as usual in the following way:

(i) $qr(\varphi) = 0$ if $\varphi$ is atomic,
(ii) $qr(\neg \varphi) = qr(\varphi),$
(iii) $qr(\varphi \land \psi) = \max(qr(\varphi), qr(\psi)),$
(iv) $qr(\exists x \varphi(x)) = qr(Qx\varphi(x)) = qr(\varphi(x)) + 1.$

$A$ and $B$ are elementary equivalent (in symbols, $A \equiv B$) if $A \equiv_n B$ for all $n$.

2. THE METRIC STRUCTURE OF $St^\tau$

Given $A, B \in St^\tau$, we define

$$d(A, B) = 1/(1 + \sup(m/A \equiv_n B)).$$

It is easy to prove that $d$ is a pseudo-metric (and then it defines an uniform topology) which generates the elementary topology of $St^r$. This topology have as basis the elementary classes (see Bell and Slomson 1969, p. 157). Moreover, since the number of formulas $\varphi$ with $qr(\varphi) \leq n$ is essentially finite, it follows that $\langle St^r, d \rangle$ is a totally bounded metric space (in fact, given $\epsilon > 0$, the collection of balls $b_\epsilon (A) = \{ B \in St^r / d (A, B) < \epsilon \}$, which is a cover of $St^r$, is finite). So, this space is compact if and only if it is metric-complete (see Kelley 1955, p. 198). A well known argument shows that the topological compactness coincide, in this case, with the logical compactness of $L^r (Q)$, where the corresponding “models” are the elements of $St^r$ (see Tarski 1952).

For example, as is known (see Bell and Slomson 1969, p. 263) that the logic $L^r (Q_0)$, where $A \models Q_0 \forall \varphi (x)$ means “there exists an infinite number of elements $a \in A$ such that $A \models \varphi [a]$”, is not compact, hence, the corresponding space $\langle St^r, d \rangle$ is not complete.

By $CSt$ we denote the class of all the Cauchy sequences of $St^r$, and by $\langle CSt, d^* \rangle$ we denote the completion of the space $\langle St^r, d \rangle$, where, for all sequences $\{ A_n \}_n$, $\{ B_n \}_n \in CSt^r$ we define

$$d^* (\{ A_n \}_n, \{ B_n \}_n) = \lim_{k \to \infty} d (A_k, B_k).$$

From this definition we have that the function $i : St^r \to CSt^r$ given by $i (A) = \{ A_n \}_n$ where $A_n = A$ for all $n$, is an isometry, $i [St^r]$ is dense in $CSt^r$ and $CSt^r$ is complete (see Kelley 1955, p. 196).

We note that the construction of the space $\langle CSt^r, d^* \rangle$ is of topological character and that the logic only appear through the initial metric $d$. In the next section we shall see a canonical way to introduce a new metric $d'$ in $CSt^r$ derived from the logic $L^r (Q)$ and equivalent to $d^*$. 

3. THE LOGICAL STRUCTURE OF $CSt^r$

Given $\{ A_n \}_n \in CSt^r$ and a sentence $\varphi \in L^r (Q)$, we say that the sequence $\{ A_n \}_n$ satisfies $\varphi$, in symbols $\{ A_n \}_n \models \varphi$, if there exists $k_0 \in N$ such that $A_k \models \varphi$ (in the usual sense) for all $k \geq k_0$. We observe that:

1. Given $\{ A_n \}_n \in CSt^r$ and $l \in N$ there exists $k_0 \in N$ such that for all $r, s \geq k_0$ we have that $A_r \equiv A_s$; hence, if $\varphi$ is a sentence of $L^r (Q)$ with $qr(\varphi) \leq l$ then either $\{ A_n \}_n \models \varphi$ or $\{ A_n \}_n \models \neg \varphi$ (and both do not holds), that is, $\{ A_n \}_n \models \varphi$ is well defined and $\{ A_n \}_n \models \neg \varphi$ iff not $\{ A_n \}_n \models \varphi$.

2. $\{ A_n \}_n \models \varphi_1 \land \varphi_2$ iff $\{ A_n \}_n \models \varphi_1$ and $\{ A_n \}_n \models \varphi_2$. 
