Squaring the Triangle*

To Professor Helmut Salzmann on the occasion of his 70th birthday

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Abstract. We construct flat Laguerre planes by ‘integrating’ flat projective planes. The construction is based in an essential way on results from the theory of interpolation. In conjunction with the unifying theory of topological circle planes and generalized quadrangles, the new construction appears to be one of the most natural and powerful constructions of such geometries.


Key words: generalized quadrangle, projective plane, circle plane, interpolation, unisolvent sets.

1. Introduction

Projective planes are much simpler structures than circle planes and generalized quadrangles. There is a variety of ways in which projective planes can be constructed from both circle planes and generalized quadrangles (cf. [12]) and it is often possible and desirable to reduce questions about the more complicated geometries to questions about the associated projective planes. On the other hand, we are interested in constructing new circle planes and generalized quadrangles. To this end, we can try to reverse the constructions that associate projective planes with these geometries. We soon find out that this is quite a hopeless undertaking, since too much information about the more complicated geometries is lost in executing these constructions.

In this paper we introduce a construction that allows us to construct many flat circle planes and three-dimensional generalized quadrangles from any given flat projective plane. We observe that the set of lines in a flat coaffine plane derived from a flat projective plane corresponds to a set of continuous functions over some interval that solves the Lagrange interpolation problem of order 2. Sets of functions like this have been extensively studied in the theory of interpolation. Straightforward integration of all functions in the set yields a set which solves the Lagrange

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interpolation problem of order 3. Even stronger, its associated geometry turns out to be a flat Laguerre plane from which a three-dimensional quadrangle can be reconstructed in a uniquely determined way.

This paper is organized as follows. In Section 2 we summarise some important information about interpolating sets of functions, flat projective planes, flat coaffine planes, three-dimensional quadrangles and flat Laguerre planes. In Section 3 we construct flat Laguerre planes by integrating interpolating sets of functions associated with flat projective planes. We also discuss how different objects associated with a flat projective plane and one of its ‘integrals’ correspond to each other and how our new construction fits in with another construction of flat Laguerre planes, or equivalently, three-dimensional quadrangles, using integration. In Section 4 we give various applications of our new construction by constructing flat Laguerre planes that admit special kinds of automorphisms. In particular, we show that all flat Laguerre planes constructed by us admit Möbius involutions. This allows us to construct flat Möbius planes from these flat Laguerre planes. If the flat projective plane we started out with admits an involutory automorphism, then it is possible to construct a Laguerre plane that also admits a Minkowski involution. This allows us to construct flat Minkowski planes from these Laguerre planes. As one further application of our construction, we construct nonclassical flat Laguerre planes that have point-transitive automorphism groups.

2. Setting the Stage

2.1. INTERPOLATING SETS OF FUNCTIONS

Let \( n \geq 1 \) be an integer, let \( I \subset \mathbb{R} \) be an interval and let \( F \) be a set of continuous functions \( I \rightarrow \mathbb{R} \). Then \( F \) is called an \( n \)-unisolvent set (of functions on \( I \)) if for any set of points \( (x_i, y_i) \in I \times \mathbb{R}, \ i = 1, \ldots, n \) with \( x_1 < x_2 \ldots < x_n \) there exists exactly one \( f \in F \) that interpolates these points, that is,

\[
    f(x_i) = y_i, \quad i = 1, \ldots, n.
\]

We remark that \( n \)-unisolvent sets are sets of functions that solve the Lagrange interpolation problem of order \( n \) which have been extensively studied in the theory of interpolation (cf. [1]).

If \( I \) is an open interval, the set \( F \) is called locally \( n \)-unisolvent if for every \( t \in I \) there is an open subinterval of \( I \) containing \( t \) such that the restriction of \( F \) to this subinterval is \( n \)-unisolvent.

We call a function \( f : [0, \pi] \rightarrow \mathbb{R} \) antiperiodic if \( f(0) = -f(\pi) \). We call a function \( g : [0, 2\pi] \rightarrow \mathbb{R} \) periodic if \( g(0) = g(2\pi) \). A set of periodic or antiperiodic functions is called \( n \)-unisolvent if it is \( n \)-unisolvent when restricted to the half-open interval \([0, \pi)\) or \([0, 2\pi)\), respectively. If \( F \) is a set of periodic functions, let the ‘set of subintervals containing the point 0’ consist of the sets \([0, 2\pi) \setminus J\), where \( J \) is a closed subinterval of the open interval \((0, 2\pi)\). Now it is also clear what it means for