HOMOCLINIC BIFURCATION
IN A PREDATOR-PREY MODEL

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1. Introduction

The starting point of this paper is the predator-prey model

\begin{equation}
N' = N \left[ \frac{\varepsilon}{K} (K - N) - \frac{u P}{\beta + N} \right], \quad P' = P \left[ -M(P) + \frac{b N}{\beta + N} \right]
\end{equation}

where \( N(t) \) and \( P(t) \) are the quantities of prey and predator, respectively; \( \varepsilon > 0 \) is the specific growth rate of prey in the absence of predation and without environmental limitation; in the absence of predators the prey population grows logistically to carrying capacity \( K > 0 \); the functional response of the predator is of Holling's type (see [6]) with satiation coefficient or conversion rate \( b \); the specific mortality of predators in absence of prey

\begin{equation}
M(P) = \frac{\gamma + \delta P}{1 + P} = \delta + \frac{\gamma - \delta}{1 + P}, \quad 0 < \gamma < \delta
\end{equation}

depends on the quantity of predators; \( \gamma \) is the mortality at low density and \( \delta \) is the maximal mortality; the natural assumption is \( \gamma < \delta \). This model was introduced by Cavani and Parkas in [2]. The advantage of the present model over the more often used models is that here the predator mortality is neither a constant nor an unbounded function, still, it is increasing with quantity. In [9] the authors made an analysis of the stability of equilibria of system (1.1) and Hopf bifurcation analysis of a nontrivial equilibrium.

In this paper, we make a bifurcation analysis of (1.1) depending on all parameters. In particular, we are interested in codimension 2 bifurcations which occur in a two dimensional parameter region. Following [7], we use singular changes of coordinates in order to transform (1.1) into a two-parameter family of ODE, where the new parameters are related to the trace-(p) and the determinant-(q) of a certain system linearized at a suitable equilibrium. We show that for each small \( q \neq 0 \), there is a locally unique \( p(q) \) for which the system (1.1) has a homoclinic orbit. Furthermore, there are parameter values for which there are periodic trajectories and other values for which (1.1) has a trajectory joining the critical points. More precisely, the homoclinic
orbit bifurcates generating a locally unique periodic orbit, which belongs to the same connected component that contains the periodic orbit generated by Andronov–Hopf bifurcation.

2. Location and stability analysis of equilibria

Under the above assumptions, the positive quadrant of the phase plane

\[ E = \{(N,P) : N \geq 0, P \geq 0\} \]

is positively invariant with respect to the system (1.1). Moreover, \( E^* = \{(N,P') : 0 \leq N \leq \Lambda, P' \geq 0\} \) is positively invariant as well, and contains the compact global attractor of the system (1.1). The only equilibria of (1.1) are two critical points \( E_1 = (0,0) \) and \( E_2 = (K,0) \) in the boundary of \( E \), which are ignored throughout the paper, and the nontrivial critical points obtained as the intersection of the curves

\[
P = f(N) := \frac{c}{aK}(K-N)(\beta + N),
\]

(2.1)

\[
P = g(N) := M^{-1}\left(\frac{bN}{\beta + N}\right) = -\frac{N-d}{N-e},
\]

(2.2)

where

\[
c = \frac{b-\gamma}{b-\delta}, \quad d = \frac{\beta \gamma}{b-\gamma}, \quad e = \frac{\beta \delta}{b-\delta}.
\]

Now, we are going to show that depending on the parameters of the system there exist at least one and at most three such nontrivial equilibria \((N,P)\).

Let us observe that from (2.1) it follows that any nontrivial equilibrium has to satisfy the condition \(0 < N < K\). Since we are interested in the case when \(0 < \gamma < \delta\), we obtain that \(b-\delta < b-\gamma\). So, we may have the following cases:

(i) \(0 < b-\delta < b-\gamma\),
(ii) \(b-\delta < b-\gamma < 0\), and
(iii) \(b-\delta < 0 < b-\gamma\).

Let us consider the case (i). A straightforward computation shows that \(c > 0\), \(0 < d < e\). Thus, if \(d < K\), then there exists just one nontrivial equilibrium. In this case the trivial equilibria are saddles. If \(d \geq K\), then there do not exist nontrivial equilibria, and \((0,0)\) is a saddle and \((K,0)\) is global asymptotically stable (see [2]), see Fig. 1.

In the case (ii), we have that \(c > 0\), \(d < e < 0\), \(e < -\beta\) and therefore the system (1.1) does not have nontrivial equilibria (see Fig. 2).

Finally, let us consider the case when \(b-\delta < 0 < b-\gamma\). This condition implies that \(c < 0\), \(e < 0 < d\) and \(e < -\beta\).

It is easy to check that \(f(N^*) = \epsilon(K + \beta)^2/4aK\), where \(N^* = (K - \beta)/2\) is the abscissa of the parabola's vertex defined by (2.1) (see Fig. 3). First, we shall show that under a suitable choice of parameters, the system (1.1)