ON THE PROJECTION OF $\tilde{N}$ ALONG THE IWASAWA DECOMPOSITION

Z. MAGYAR (Budapest)

Abstract. Let $G = KAN$ be the Iwasawa decomposition of a real Lie group $G$ with reductive Lie algebra $\mathfrak{g}$. If $\kappa : G \to K$ is the corresponding projection and $N$ is the usual “other $N$ made with the opposite positivity” then we show $\kappa(\tilde{N})$ has compact closure even though $K$ is not necessarily compact.

Consider a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ which is reductive, i.e., the direct product of a commutative Lie algebra and simple Lie algebras. Some of the real Lie groups $G$ having Lie algebra isomorphic to $\mathfrak{g}$ may be called reductive; various authors have considered somewhat different classes. The present author proposed after Corollary 2.3 of [4] a class which contains the others and is still subject to the usual theory.

If $G$ is any real Lie group with reductive Lie algebra $\mathfrak{g}$ then we can certainly develop the semi-simple framework for it as described in the introduction of [4]; we call attention that in our terminology $\mathfrak{g}$ always contains the whole center $\mathfrak{z}$ of $\mathfrak{g}$. As in the present note we are not interested in nonminimal parabolics, we shall simply denote by $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{N}$, $\mathfrak{N}$ what is denoted in [4] by $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{A}$, $\mathfrak{M}$, $\mathfrak{N}$ what is denoted in [4] by $\mathfrak{a}$, $\mathfrak{m}$, $\mathfrak{n}$, $\mathfrak{A}$, $\mathfrak{M}$, $\mathfrak{N}$, resp. If $g = kan$ with $k \in K$, $a \in A$ and $n \in N$ then we put

(1) \[ \kappa(g) := k. \]

Let $\tilde{N} := \exp(\mathfrak{m} \mathfrak{n})$; it is known that $\tilde{N}$ can in a natural way be identified with an open subset of the compact homogeneous space $G/(MAN)$; and the closure of this open set is a connected component of $G/(MAN)$. In particular, if the outer Weyl group $OW$ of $G$ is trivial (see the Introduction of [4] for this concept), which is usually assumed by other authors, then $\tilde{N}$ is dense in $G/(MAN)$, and its complement has lower dimension in a certain sense. Therefore in many arguments about parabolically induced representations $\tilde{N}$ can be used instead of $G/(MAN)$; when one does so then it is usually said “we apply the noncompact picture”. In the so called compact picture $G/(MAN)$ is identified with $K/M$, and therefore $\kappa|M$ is the mapping providing the link between the compact and noncompact pictures. So the

* research supported, in part, by the Hungarian National Fund for Scientific Research, grant numbers 17435 and 16924.

0236–5294/94/$5.00 (c) 1994 Akadémiai Kiadó, Budapest
"boundedness" of $\kappa(\overline{N})$ may have a lot of applications. Of course, the connected Lie subgroup $H$ with Lie algebra $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ contains both $\mathfrak{AN}$ and $\mathfrak{N}$, thus $\kappa(\mathfrak{N}) \subset \mathfrak{H}$; so in studying $\kappa|_{\mathfrak{N}}$ we may switch to $H$. In other words, we may and shall assume in the sequel that $\mathfrak{g}$ is semi-simple and $G$ is connected.

Unfortunately, the above restrictions do not imply that $K$ is compact: if $\mathfrak{k}$ is not semi-simple then e.g. for simply connected $G$ its (simply connected) $K$ is not compact. For semi-simple $\mathfrak{g}$ we have this if and only if $\mathfrak{g}$ contains a simple ideal of type $\mathfrak{su}(p,q)$, $\mathfrak{so}(p,2)$, $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}^*(2n)$, $\mathfrak{e}_6(-14)$ or $\mathfrak{e}_7(-25)$. However, it is well known that the center of $\mathfrak{k}$ splits into one dimensional subspaces along these ideals of $\mathfrak{g}$. We shall not rely on these facts, but they make it clearer what we shall talk about. We emphasize that the problem already arises at the smallest noncompact simple real Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{su}(1,1)$: then $\mathfrak{k} \simeq \mathbb{R}$ as a Lie algebra, and for the corresponding simply connected $G$ we have $K \simeq (\mathbb{R},+)$ as Lie groups.

Since $K/M$ is compact and with the projection $p : K \twoheadrightarrow K/M$ we have that $p \circ \kappa|_{\mathfrak{N}}$ is an analytic imbedding, one might guess that $\kappa(\overline{N})$ is "bounded", i.e., has compact closure even if $K$ is not compact. We shall see this is the case indeed.

Recall some details of the Iwasawa decomposition. We can find a scalar product over $\mathfrak{g}$ and an ordered orthonormal basis with respect to it such that $\text{ad} \varphi = -(\text{ad} \varphi)^*$ for all $\varphi \in \mathfrak{g}$, and any matrix in ad $a$ and in ad $n$ becomes diagonal and strictly upper triangular, respectively. Let $N(\mathfrak{g})$ and $N(\mathfrak{g})$ be the "A" and "N" of $S\mathfrak{L}^\circ(\mathfrak{g})$, resp., i.e. let $A(\mathfrak{g})$ be the group of diagonal matrices whose entries are positive and have product 1, and let $N(\mathfrak{g})$ be the group of upper triangular matrices with 1's in the diagonal. If $\kappa_*$ is the "$N" of S\mathfrak{L}^\circ(\mathfrak{g})$ with respect to its Iwasawa decomposition $S\mathfrak{L}^\circ(\mathfrak{g}) = A(\mathfrak{g})N(\mathfrak{g})$, then

$$\text{Ad} \circ \kappa = \kappa_* \circ \text{Ad}.$$  

We note that if one uses the greater group $\text{GL}(\mathfrak{g})$ (whose "$K" is $\text{O}(\mathfrak{g}) \times (0, +\infty) \cdot I$) then (2) also holds for nonconnected $G$ (we shall not use this fact).

Fix $x \in \mathfrak{g}$ and write $X = \text{ad} x$; then $X$ is a strictly lower triangular matrix. We should obtain some "bound" for $\kappa(\exp x)$ depending only on $G$; in fact, we shall get an explicit bound expressed in terms of $\mathfrak{g}$. Put $\gamma(t) := \kappa(\exp tx)$ for $t \in \mathbb{R}$; this is an analytic curve in $K$, and from (2) we infer

$$\text{Ad} \circ \gamma(t) = \kappa_* e^{tX}.$$  

This formula, the continuity of $\gamma$ and the fact $\gamma(0) = 1$ together determine $\gamma$ because $\text{Ad}$ is a covering for semi-simple $\mathfrak{g}$.

**Lemma 1.** The equation $\det(I + \kappa_*(e^{tX})) = 0$ has just a finite number of solutions in $t$; moreover, this number is bounded by a function of $\nu = \dim \mathfrak{g}$ (so if $X$ and even the Lie algebra structure of $\mathfrak{g}$ varies then we still find a common bound).