THE WEIGHTED LEBESGUE CONSTANT OF
LAGRANGE INTERPOLATION FOR
EXPONENTIAL WEIGHTS ON $[-1, 1]$

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Dedicated to Prof. Mario Rosario Occorso on his 65th birthday

Abstract. We establish uniform estimates for the weighted Lebesgue constant of Lagrange interpolation for a large class of exponential weights on $[-1, 1]$. We deduce theorems on uniform convergence of weighted Lagrange interpolation together with rates of convergence.

1. Introduction

In this paper, we investigate the Lebesgue function and Lebesgue constant of Lagrange interpolation for weights $w := \exp(-Q)$ where $Q : (-1, 1) \to \mathbb{R}$ is even, convex in $(-1, 1)$ and grows sufficiently rapidly and smoothly near $\pm 1$. Classical examples of these weights are:

(1.1) $w_{0, \alpha}(x) := \exp\left(-\left(1 - x^2\right)^{-\alpha}\right), \quad \alpha > 0$

(1.2) $w_{k, \alpha}(x) := \exp\left(\exp_k\left(1 - x^2\right)^{-\alpha}\right), \quad \alpha > 0, \quad k \geq 1.$

Here $\exp_k := \exp\left(\exp(\ldots)\right)$ denotes the $k$th iterated exponential.

We note that for $\alpha \geq \frac{1}{2}$, $w_{0, \alpha}$ violates Szegő’s condition,

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1 - x^2}} dx > -\infty.$$ 

We are interested in approximating continuous functions $f : (-1, 1) \to \mathbb{R}$ by weighted polynomials $P_n w$ of degree at most $n$, $n \geq 1$ in the uniform norm and to this end it will be necessary to impose a decay condition on the given function $f$. More precisely, we will suppose henceforth that our given $f$ satisfies

(1.3) \[ \lim_{|x| \to 1} |f w(x)| = 0. \]

* The research of this paper was completed while the author was visiting the University of South Florida during the Fall Semester, 1996.
The following important notation will be used in the sequel. Let $\mathcal{P}_n$ denote the class of algebraic polynomials of degree at most $n$ and set

\begin{equation}
E_n[f]_{w,\infty} := \inf_{P \in \mathcal{P}_n} \| (f - P)(x)w(x) \|_{L_{\infty}[-1,1]}.
\end{equation}

This quantity is the error of best weighted polynomial approximation to $f$ from $\mathcal{P}_n$, $n \geq 1$ and it is well known [8] that

\[ E_n[f]_{w,\infty} \to 0 \text{ as } n \to \infty. \]

For our approximation we will use weighted Lagrange interpolation operators and to this end we let

\[ \chi_n := \{\xi_{1,n}, \xi_{2,n}, \ldots, \xi_{n,n}\}, \quad n \geq 1 \]

be an arbitrary set of nodes in $[-1,1]$. The Lagrange interpolation polynomial to $f$ with respect to $\chi_n$ is denoted by $L_n[f, \chi_n]$. Thus, if $l_{j,n}(\chi_n) \in \mathcal{P}_{n-1}$, $1 \leq j \leq n$ are the fundamental polynomials of Lagrange interpolation at the $\xi_j$, $1 \leq j \leq n$ satisfying $l_{j,n}(\chi_n)(\xi_k,n) = \delta_{j,k}$ for $1 \leq k \leq n$ then it is well known that

\begin{equation}
L_n[f, \chi_n](x) := \sum_{j=1}^{n} f(\xi_j,n) l_{j,n}(\chi_n)(x) \in \mathcal{P}_{n-1}.
\end{equation}

It is customary to write

\begin{equation}
\| w(f - L_n[f, \chi_n]) \|_{L_{\infty}[-1,1]} \leq E_{n-1}[f]_{w,\infty} \left(1 + \| w(x) \sum_{j=1}^{n} |l_{j,n}(\chi_n)(x)| w^{-1}(\xi_j,n) \|_{L_{\infty}[-1,1]}\right)
\end{equation}

\[ := E_{n-1}[f]_{w,\infty} \left(1 + \Lambda_n(w, \chi_n)\right). \]

Here $\Lambda_n(w, \chi_n)$ is called the Lebesgue constant with respect to the weight $w$ and the set of nodes $\chi_n$, and $\lambda_n(w, \chi_n, x)$ is the corresponding Lebesgue function.

Using (1.6), it is well known and easy to see that estimates of the size of the Lebesgue constant and the error $E_{n-1}[f]_{w,\infty}$ yield theorems on uniform convergence of Lagrange interpolation.

Recently, Szabados in his paper [12] investigated the order of the weighted Lebesgue constant for Freud weights on $\mathbb{R}$. His methods were further explored by the author in [1] for Erdős weights on $\mathbb{R}$. These papers laid the