EXAMPLES OF 4-MANIFOLDS WITHOUT GOMPF NUCLEI

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Abstract. A problem of Kirby (Problem 4.98 in [9]) will be answered in the negative. We show that the 4-manifold $X_{2,2,2}$ defined below does not contain the Gompf nucleus $N_2$. More generally, we also show the existence of 4-manifolds without Gompf nuclei $N_n$. The proofs rely on the connection between the smooth topology and the Seiberg–Witten basic classes of a given 4-dimensional manifold $M$.

1. Introduction

We devote this note to answer a question in Kirby’s problem list [9] asserting whether every simply connected smooth 4-manifold with $b^+ \geq 3$ contains a Gompf nucleus $N_2$ (Problem 4.98 in [9]). We prove that by doing logarithmic transformations on three linearly independent tori in the $K3$-surface we get a 4-manifold not containing $N_2$. To make our statements more precise we need a few definitions.

The hypersurface $X = \{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 | \sum_{i=0}^3 z_i^4 = 0 \} \subset \mathbb{CP}^3$ is a simply connected, smooth 4-manifold with $c_1(X) = 0$, hence it is a $K3$-surface. It is known that all simply connected complex surfaces with vanishing first Chern class are diffeomorphic [2], consequently from the differential topological point of view $X$ is the $K3$-surface. The complex surface $X$ admits a holomorphic fibration $\pi : X \to \mathbb{CP}^1$ such that the generic fiber is a smooth elliptic curve — a 2-dimensional torus. Such fibrations are called elliptic fibrations. Note that $\pi : X \to \mathbb{CP}^1$ is not a fiber bundle, it might have singular fibers as well. It can be assumed that $\pi$ has at least one singular fiber homeomorphic to the 2-dimensional sphere $S^2$ — such fibers are called cusp fibers. The tubular neighborhood of a cusp fiber admits a handle decomposition with a single 0-handle and a 2-handle attached to $S^3 = \partial(0$-handle) according to the Kirby diagram of Fig. 1. The fibration also has a section $\sigma : \mathbb{CP}^1 \to X$, i.e., a map such that for every value $t \in \mathbb{CP}^1$ the intersection $\sigma(\mathbb{CP}^1) \cap \pi^{-1}(t)$ consists of one point. The image $\sigma(\mathbb{CP}^1)$ is a sphere $S \subset X$, and one can prove that the square of the homology class of $[S]$ is equal to $-2$. The Gompf nucleus $N_2$ is by definition the tubular neighborhood of the union of a cusp fiber and a section in $X$. The manifold $N_2$ admits a handle decomposition with one 0-handle and two 2-handles, where these two 2-handles are attached to $S^3 = \partial(0$-handle) according to the Kirby dia-
gram shown in Fig. 2. It can be shown that the $K3$-surface $X$ contains three disjoint copies of $N_2$ [7] (corresponding to three different elliptic fibrations of the smooth 4-manifold $X$). If a 4-manifold $M^4$ contains a 2-dimensional torus $T$ with square 0, then we can perform a logarithmic transformation on $T$: deleting the tubular neighborhood of $T$ (which is diffeomorphic to the product $D^2 \times T^2$ of the 2-dimensional torus $T^2$ with the 2-disk $D^2$) and regluing it via a diffeomorphism $\varphi : \partial (M \setminus D^2 \times T^2) \to \partial (D^2 \times T^2)$ we get a new manifold $M_\varphi$. It turns out that if $T$ is the fiber in a Gompf nucleus $N_2 \subset M$, then the diffeomorphism type of $M_\varphi$ depends only on one non-negative number $p$ associated to $\varphi$; this number is called the multiplicity of the logarithmic transformation. For more about elliptic surfaces, nuclei and logarithmic transformations see [3], [6] or [8].

Now perform logarithmic transformations of multiplicity 2 on the three tori contained by the three disjoint nuclei in the $K3$-surface $X$. The resulting manifold will be denoted by $X_{2,2,2}$. 

Acta Mathematica Hungarica 83, 1999