ON HOPFIAN RINGS

K. VARADARAJAN (Calgary)

Abstract. The main results proved in this paper are:
(i) If \( R \) is a boolean hopfian ring then the polynomial ring \( R[T] \) is hopfian.
(ii) Let \( R \) and \( S \) be hopfian rings. Suppose the only central idempotents in \( S \) are 0 and 1 and that \( S \) is not a homomorphic image of \( R \). Then \( R \times S \) is a hopfian ring.

1. Introduction

Throughout this paper the rings we consider will be associative rings with \( 1 \neq 0 \) and ring homomorphisms will be required to preserve the identity elements. A map \( f : R \to S \) between rings preserving addition and multiplication but not necessarily satisfying \( f(1_R) = 1_S \) will be referred to as a not necessarily unital ring homomorphism. We write \( R \cdot M \) to denote that \( M \) is a unital left \( R \)-module. Given a ring homomorphism \( f : R \to S \), an additive map \( \varphi : R \cdot M \to S \cdot N \) will be called \( f \)-semilinear if \( \varphi(rx) = f(r)\varphi(x) \) for any \( x \in M \) and \( r \in R \). When we talk of a subring \( R' \) of \( R \) we require \( 1_{R'} = 1_R \).

Definition 1. A ring \( R \) will be said to be hopfian if every surjective ring homomorphism \( f : R \to R \) is automatically an isomorphism.

Let \( R \) be a ring. If \( R \) satisfies a.c.c for two-sided ideals then \( R \) is a hopfian ring (Proposition 1.9 (i) in [4]). In particular any left or right noetherian ring; any ring with finitely many two-sided ideals (for instance any simple ring) will be hopfian. One of the proofs of Hilbert's Basis Theorem goes over word for word to yield the result that the polynomial ring \( R[T] \) satisfies a.c.c for two-sided ideals whenever \( R \) does. In particular if \( R \) has a.c.c for two-sided ideals, the ring \( R[T] \) is hopfian. From these observations one is led to the question whether there are commutative hopfian rings which are not noetherian.

Recall that \( R \) is called a boolean ring if \( x^2 = x \) for every \( x \in R \). It is well known that any boolean ring \( R \) is commutative and that \( 2x = 0 \) for any \( x \in R \). If \( X \) is a compact totally disconnected Hausdorff space, the collection \( B(X) \) of clopen subsets of \( X \) is a boolean ring under the operations \( C + D = (C \cup D) \setminus (C \cap D) \) and \( C \cdot D = C \cap D \) for any \( C, D \) in \( B(X) \). A well known result of M. H. Stone (see Appendix 3 of [3]) asserts that any boolean ring \( B \) is isomorphic to the boolean ring of clopen subsets of a compact totally dis-
connected Hausdorff space $X_B$, determined uniquely up to homeomorphism by the isomorphism class of $B$. $X_B$ will be referred to as the Stone space of $B$.

We will adopt the convention that all spaces considered are Hausdorff.

**Definition 2.** A topological space $X$ is said to be co-hopfian if every injective self continuous map $f : X \to X$ is automatically a homeomorphism.

It was shown in [4] that a boolean ring $B$ is hopfian if and only if its Stone space $X_B$ is co-hopfian (Theorem 4.4, [4]). In [2] Satya Deo and the present author have proved the existence of at least $2^c$ non-homeomorphic perfect compact totally disconnected co-hopfian spaces where $c$ is the cardinality of the set of real numbers. In Section 2 of the present paper, using M. H. Stone's representation theorem we will show that a boolean ring $B$ is noetherian if and only if it is finite. It follows immediately that if $X$ is an infinite compact totally disconnected co-hopfian space then $B(X)$ is a commutative non-noetherian hopfian ring. It will be interesting to find an example of a hopfian commutative non-noetherian integral domain.

The problem of deciding whether the hopficity of the ring $R$ implies the hopficity of the ring $R[T]$ is a basic question in ring theory. In Section 3 we obtain partial results concerning this question. Let $R$ and $S$ be hopfian rings. The question of determining whether $R \times S$ is hopfian as a ring is again a basic question. In Section 4 we obtain partial results on this problem.

**2. Noetherian boolean rings**

In this section $X$ denotes a compact totally disconnected space and $B(X)$ the boolean ring of clopen subsets of $X$.

**Proposition 1.** If $X$ is infinite we can find an infinite sequence $\{a_j\}_{j \geq 1}$ of elements in $B(X)$ satisfying $B(X)a_j \subset B(X)a_{j+1}$ for all $j \geq 1$, where $\subset$ denotes strict inclusion.

**Proof.** Since $X$ is an infinite compact space, there exists a non-isolated point $x \in X$. Since clopen sets form a base for the topology of $X$ we can find an infinite sequence $\{C_k\}_{k \geq 1}$ of clopen subsets of $X$ satisfying $x \in C_k$ and $C_{k+1} \subset C_k$ for all $k \geq 1$. Then $F_k = X \setminus C_k$ is a sequence of clopen sets of $X$ satisfying $F_k \subset F_{k+1}$ for all $k \geq 1$. Moreover $F_k = F_k \cap F_{k+1}$ and $F_{k+1} \neq D \cap F_k$ for any clopen subset $D$ of $X$. Writing $a_k$ for the element of $B(X)$ corresponding to the clopen set $F_k$ of $X$ we have $a_k = a_k a_{k+1}$ and $a_{k+1} \not\in B(X)a_k$ for any $k \geq 1$. Thus the sequence $\{a_j\}_{j \geq 1}$ satisfies the requirements of Proposition 1. \qed

The following well-known result is an immediate consequence of Proposition 1. The usual proofs are purely algebraic. Our proof which relies on

*Acta Mathematica Hungarica 83, 1999*