A CLASS OF OPERATORS ON $L^2_D$. II

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Abstract. Let $D$ be the open unit disk in $\mathbb{C}$, and $L^2_D$ the space of quadratic integrable, harmonic functions defined on $D$. Let $\varphi : \overline{D} \to \mathbb{C}$ be a function in $L^\infty(D)$ with the property that $\varphi(b) = \lim_{r \to 0, x \in \delta D} \varphi(x)$ for all $b \in \partial D$. Define the operator $C_\varphi$ on $L^2_D$ as follows: $C_\varphi(f) = Q(\varphi \cdot f)$, where $Q$ is the orthogonal projection of $L^2(D)$ onto $L^2_D$. In this paper it is shown that if $C_\varphi$ is Fredholm, then $\varphi$ is bounded away from zero on a neighborhood of $\partial D$. Also, if $C_\varphi$ is compact, then $\varphi(\partial D) \equiv 0$, and the commutator ideal of $\tau(D)$ is $K(D)$, where $\tau(D)$ denotes the norm closed subalgebra of the algebra of all bounded operators on $L^2_D$ generated by $\{ C_\psi : \psi \in C(\overline{D}) \}$, and $K(D)$ is the ideal of compact operators on $L^2_D$.

Finally, the spectrum of classes of operators defined on $L^2_D$ is characterized.

Introduction

$L^2_D$ is the space of all harmonic functions $f$ defined on the open unit disk $\overline{D}$ such that they are square integrable with respect to the area measure $dA = \frac{1}{\pi} dydx$. Following similar arguments to that used by Conway [3, p. 175] it can be shown that $L^2_D$ is a closed subspace of $L^2(D)$ with orthonormal basis $\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}z, \sqrt{\frac{3}{\pi}}z^2, \ldots$. Moreover, it can be easily established that for each $\lambda \in D$ there is a unique $k_\lambda$ in $L^2_D$ such that $f(\lambda) = \langle f, k_\lambda \rangle$. $f \in L^2_D$, $k_\lambda$ is the reproducing kernel of $L^2_D$. The Bergman space $A^2$ is the space of all analytic functions $f$ defined on $D$ and square integrable with respect to the area measure. It is known that $A^2$ is a closed subspace of $L^2(D)$ with orthonormal basis $\{ \sqrt{n+1}z^n \}_{n \geq 0}$. The space $L^2_D = A^2 \oplus \overline{A}_0$, where $\overline{A}_0$ is the space of all complex conjugates of functions in $A^2$ which vanish at the origin.

Let $\varphi \in L^\infty(D)$ and $Q$ be the orthogonal projection of $L^2(D)$ onto $L^2_D$. Define the operator $C_\varphi$ on $L^2_D$ by $C_\varphi(f) = Q(\varphi \cdot f)$. It can be easily established that $C_{\alpha \varphi + \beta \psi} = C_\alpha C_{\varphi}$, $\alpha, \beta \in \mathbb{C}$, and $C_\varphi^* = C_{\overline{\varphi}}$, where $C_\varphi^*$ is the adjoint of $C_\varphi$, $\overline{\varphi}$ is the complex conjugate of $\varphi, \psi \in L^\infty(D)$.

In Section 1 of this paper, it is shown that if $\varphi \in L^\infty(D)$ with the property that $\varphi(b) = \lim_{r \to 0, x \in \delta D} \varphi(x)$ for all $b \in \delta D$ (so that $\varphi(\partial D)$ is continuous), and $C_\varphi$ is Fredholm, then $\varphi$ is bounded away from zero on a neighborhood of $\partial D$. Also, if $C_\varphi$ is compact, then $\varphi(\partial D) \equiv 0$. The converse of the above results appeared in [5]. Finally, it is proved in this paper that the commutator
ideal of \( \tau(D) \) is \( K(D) \), where \( \tau(D) \) denotes the norm closed subalgebra of the algebra of all bounded operators on \( L^2_h \) generated by \( \{ C_\psi : \psi \in C(D) \} \), and \( K(D) \) is the ideal of compact operators on \( L^2_h \). Similar results concerning Toeplitz operators defined on the Bergman space \( A^2 \) appeared in [2], and [8]. In Section 2, the spectrum of classes of operators defined on \( L^2_h \) is determined.

1. Results

**Theorem 1.1.** Let \( \varphi : \overline{D} \to \mathbb{C} \) be a function in \( L^\infty(D) \) with the property that \( \varphi(b) = \lim_{r \to 0} \frac{\varphi(rz) + \varphi(r \bar{z})}{2} \) for all \( b \in \partial D \). If \( C_\varphi \) is Fredholm, then \( \varphi \) is bounded away from zero on a neighborhood of \( \partial D \).

To prove Theorem 1.1 the following is needed.

**Lemma 1.1.** Let \( k_\alpha \) be the reproducing kernel of \( L^2_h \). Then

\[
k_\alpha(z) = 2 \text{Re} \left( (1 - \bar{\alpha}z)^{-2} \right) - 1, \quad \alpha, z \in D.
\]

**Proof.** Let \( \{ e_n \}_{n \geq 0} = \{ \sqrt{n + 1} z^n \} \) and \( \{ e_n \}_{n \geq 1} = \{ \sqrt{n + 1} \alpha^n \} \). Utilizing the facts that \( L^2_h \) is a separable Hilbert space, and \( \ldots, \sqrt{3 \pi^2}, \sqrt{2 \pi}, \sqrt{\pi}, \sqrt{2}, \sqrt{3}, \ldots \) is an orthonormal basis of \( L^2_h \), it follows that

\[
k_\alpha = \sum_{n \geq 0} (k_\alpha, e_n) e_n + \sum_{n \geq 1} (k_\alpha, \bar{e}_n) \bar{e}_n.
\]

Note that for \( n \geq 0 \), \( (k_\alpha, e_n) = \sqrt{n + 1} \pi^n \), and for \( n \geq 1 \), \( (k_\alpha, \bar{e}_n) \) is equal to \( \sqrt{n + 1} \alpha^n \). Thus,

\[
k_\alpha(z) = \sum_{n \geq 0} (n + 1)(\alpha z)^n + \sum_{n \geq 1} (n + 1)(\alpha \bar{z})^n = (1 - \alpha z)^{-2} - (1 - \alpha \bar{z})^{-2} - 1 = 2 \text{Re} \left( (1 - \bar{\alpha}z)^{-2} \right) - 1. \quad \square
\]

Let \( \{ \lambda_n \}_{n \geq 1} \) be a sequence in \( D \) which converges to some \( \lambda \in \partial D \). For each \( m \), define

\[
f_m(z) = k_{\lambda m}(z)/(k_{\lambda m}(\lambda_m))^{\frac{1}{2}}
\]

*Acta Mathematica Hungarica* 84, 1999