ON THE ISOSELES ORTHOGONALLY EXPONENTIAL MAPPINGS

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Abstract. We prove that if $X$ is a real linear normed space and $\dim X > 1$, then, for every isosceles orthogonally exponential mapping $f$ of $X$ into a division ring, either $f(X \setminus \{0\}) = \{0\}$ or $0 \not\in f(X)$. As a consequence of this fact we obtain the following theorem: If $X$ is not an inner product space and $\dim X > 2$, then every isosceles orthogonally exponential mapping of $X$ into a (commutative) field is exponential. We also generalize some results concerning the orthogonally additive mappings.

Let $X$ be a real normed space. Define a binary relation $\perp \subset X^2$ by

$$x \perp y \quad \text{if} \quad ||x + y|| = ||x - y||.$$

The relation $\perp$ is called the isosceles orthogonality (see [10] and [11]).

Let $K$ be a ring. A mapping $f : X \to K$ is called exponential if

(1) $f(x + y) = f(x)f(y)$ for every $x, y \in X$;

it is called isosceles orthogonally exponential if

(2) $f(x + y) = f(x)f(y)$ whenever $x \perp y$.

We start with some lemmas.

**Lemma 1.** Let $X$ be a real linear normed space with $\dim X > 1$. Then, for every $x \in X$, there exists $y \in X$ with $||x|| = ||y||$ and $x \perp y$.

**Proof.** We argue in the same way as in the proof of Theorem in [10] (p. 270). □

**Lemma 2.** Let $X$ be as in Lemma 1, $K$ a division ring, and $f : X \to K$ a solution of (2). Then the following four conditions hold:

(i) If $f(x) \neq 0$, then $f\left(\frac{1}{2}y\right) f\left(-\frac{1}{2}y\right) \neq 0$ for every $y \in X$ with $||x|| = ||y||$ and $x \perp y$.

(ii) If $f(x) \neq 0$, then $f\left(\frac{1}{2}x\right) \neq 0$.

(iii) If $f(x) \neq 0$, then $f(-x) \neq 0$.

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(iv) If \( f(x) \neq 0 \), then \( f(2x) \neq 0 \).

**Proof.** Fix \( x \in X \) such that \( f(x) \neq 0 \). The case \( x = 0 \) is trivial. So suppose that \( x \neq 0 \).

(i), (ii) Take \( y \in X \) with \( \|x\| = \|y\| \) and \( x \perp y \). Then \( \frac{1}{2}(x+y) \perp \frac{1}{2}(x-y) \) and, by (2),

\[
f(x) = f \left( \frac{1}{2}(x+y) + \frac{1}{2}(x-y) \right) = f \left( \frac{1}{2}x \right) f \left( \frac{1}{2}y \right) f \left( \frac{1}{2}x \right) f \left( -\frac{1}{2}y \right).
\]

Hence \( f \left( \frac{1}{2}y \right) f \left( -\frac{1}{2}y \right) f \left( \frac{1}{2}x \right) \neq 0 \). Thus Lemma 1 completes the proof.

(iii) On account of (ii), \( f \left( \frac{1}{2}x \right) \neq 0 \). According to Lemma 1 there is \( y \in X \) with \( \|x\| = \|y\| \) and \( x \perp y \). Note that (i) and (ii) imply

\[
f \left( \frac{1}{2}y \right) f \left( -\frac{1}{2}y \right) f \left( \frac{1}{4}y \right) f \left( -\frac{1}{4}y \right) \neq 0
\]

and consequently, by (i), \( f \left( -\frac{1}{4}x \right) \neq 0 \) (because \( \frac{1}{2}y = \|\frac{1}{2}x\| \) and \( \frac{1}{2}y \perp \frac{1}{2}x \)). Thus

\[
f \left( -\frac{1}{2}x \right) = f \left( \frac{1}{4}(-x-y) + \frac{1}{4}(-x+y) \right)
\]

\[
= f \left( -\frac{1}{4}x \right) f \left( -\frac{1}{4}y \right) f \left( -\frac{1}{4}x \right) f \left( \frac{1}{4}y \right) \neq 0.
\]

Hence

\[
f(-x) = f \left( -\frac{1}{2}x \right) f \left( -\frac{1}{2}y \right) f \left( -\frac{1}{2}x \right) f \left( \frac{1}{2}y \right) \neq 0.
\]

(iv) Take \( y \) as before. Then, by (i)–(iii) and (2),

\[
f(y) = f \left( \frac{1}{2}(y+x) + \frac{1}{2}(y-x) \right) = f \left( \frac{1}{2}y \right) f \left( \frac{1}{2}x \right) f \left( \frac{1}{2}y \right) f \left( -\frac{1}{2}x \right) \neq 0.
\]

Thus, according to (iii), \( f(-y) \neq 0 \) and consequently

\[
f(2x) = f(x+y+x-y) = f(x)f(y)f(x)f(-y) \neq 0.
\]

**Lemma 3.** Let \( X, K \) and \( f \) be as in Lemma 2. Suppose that \( x \in X \) and \( f(x) = 0 \). Then \( f(z) = 0 \) for every \( z \in X \) with \( \|z\| = \|x\| \).

**Proof.** Take \( z \in X \) with \( \|z\| = \|x\| \). According to Lemma 2(ii), \( f(2x) = 0 \). Further, \( x + z \perp x - z \) and \( x + z \perp z - x \). Thus, by (2),

\[
f(x+z)f(x-z) = f(2x) = 0, \quad f(2z) = f(x+z)f(z-x).
\]

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