G-UNIFORMITIES, LR-PROXIMITIES AND HYPERTOPOLOGIES

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Dedicated to Professor Á. Császár on the occasion of his 75th birthday

Abstract. We look closely at the relationships between hit-and-miss and proximal hit-and-miss Δ-topologies, in the setting of proximity spaces. We provide equivalent conditions that force comparisons among proximal hit-and-miss Δ-topologies determined by different proximities. We pay attention to these topologies when Δ consists of the family of all closed balls of a proximity space, and we study their interplay with the Wijsman convergence expressed in proximity spaces. Finally we study the supremum of all Wijsman convergences and of all proximal ball topologies when X is at most regular, and the infimum of all Wijsman convergences when X is at least Tychonoff.

1. Introduction

Hyperspace topologies, i.e. topologies on the set of all nonempty closed subsets of a topological space X, were first studied in a systematic way by Michael [22] in 1951. Later, Poppe [27] considered abstract hit-and-miss Δ-topologies, whereas, Nachman [25] studied their proximal analogues. There is now a rich literature devoted to the study of hit-and-miss and proximal hit-and-miss Δ-topologies ([1], [3], [4], [8], [9], [10], [14], [19], [20], [31], [32]). Following [3], [9], [11], [12], [13], [20], we continue in the study of comparison of hyperspace topologies. More precisely, we look for necessary and sufficient conditions for the pairwise coincidence of various Δ-and proximal Δ-topologies, in the context of proximity spaces. Since X can have an infinite spectrum of proximities compatible with respect to its topology, it is possible to have an unusually broad range of proximal hit-and-miss Δ-topologies corresponding to an abstract hit-and-miss Δ-topology. We provide a careful analysis of the relationships that exist between two proximal Δ-topologies associated to different compatible proximities on X. Using the tool of generalized uniform structures we extend the notion of ball, proximal ball and Wijsman convergence in the setting of proximity spaces, whose induced topologies are at most regular. We look for conditions under which ball topology, proximal ball topology and Wijsman convergence are pairwise equal. We also compare Wijsman convergences associated with different generalized uniform structures. We get some interesting characterizations for the coincidence, from which it is possible to derive, as nice corollaries, some central theorems stated in [3], [5], [11] and [20].

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We point out that although we are not interested in the admissibility of the hypertopologies, neither in conditions on \( X \) that assure it (i.e. \( T_1 \) axiom), we obtain results that still hold in the context of the admissibility introduced by Michael [22].

We study the supremum of all proximal ball topologies obtaining results similar to those in [2], when the proximity on the base space is only \( (LR) \). If we consider the Wijsman convergences built on compatible generalized uniform structures, then we distinguish two cases:

1) if they give a fixed compatible \( (LR) \)-proximity on \( X \), then, in general, the supremum is different from proximal topology;

2) if they give all compatible \( (LR) \)-proximities on \( X \), then the supremum is the Vietoris topology.

Finally, we extend Theorem 2.4 in [10] to the uniform case, showing that the infimum of all Wijsman convergences coincides with the proximal totally bounded topology. We show that the same result does not hold for generalized uniform structures compatible with \( (LR) \)-proximities.

2. Preliminaries

Let \( (X, \tau) \) be a topological space with no separation property. For any \( E \subset X, \bar{E}, \text{int} E \) and \( E^c \) stand for the closure, interior and complement of \( E \) in \( X \), respectively. Let \( \alpha \) be a binary relation on the power set of \( X \). Consider the following axioms:

(i) \( A\alpha B \) implies \( B\alpha A \);

(ii) \( A\alpha (B \cup C) \) implies \( A\alpha B \) or \( A\alpha C \);

(iii) \( A\alpha B \) implies \( A \neq \emptyset, B \neq \emptyset \);

(iv) \( A \cap B \neq \emptyset \) implies \( A\alpha B \);

(v) \( A\alpha B \) and \( b\alpha C \) for every \( b \in B \) together imply \( A\alpha C \);

(vi) \( a\not\in B \) implies there exists \( E \subset X \) such that \( a\not\in E \) and \( E^c \not\in B \);

(vii) \( a\not\in B \) implies there exists \( E \subset X \) such that \( A\not\in E \) and \( E^c \not\in B \).

The relation \( \alpha \) is called an \( (LO) \)-proximity iff it satisfies (i)-(iv) and (v) (see [26]); an \( (R) \)-proximity iff it satisfies (i)-(iv) and (vi) (see [17]); an \( (EF) \)-proximity iff it satisfies (i)-(iv) and (vii) (see [26]); an \( (LR) \)-proximity iff it is simultaneously an \( (LO) \)- and an \( (R) \)-proximity (see [18]).

By a proximity \( \alpha \) we mean any one of the above proximities. Let \( \alpha \) be an \( (LO) \)- or \( (R) \)-proximity on \( X \). Denote by \( \tau(\alpha) \) the topology on \( X \) induced by the Kuratowski closure operator \( A \mapsto A^\alpha = \{ x \in X : x\alpha A \} \). \( \alpha \) is compatible with respect to the topology \( \tau \) on \( X \) iff \( \tau = \tau(\alpha) \) (see [21], [26]). If \( \alpha \) is an \( (LO) \)-proximity on \( X \), then (see [23], [26], [12]):

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