ASYMPTOTIC DENSITY AND THE ASYMPTOTICS OF PARTITION FUNCTIONS

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Abstract. Let $A$ be a set of positive integers with $\gcd(A) = 1$, and let $p_A(n)$ be the partition function of $A$. Let $c_0 = \pi \sqrt{2/3}$. If $A$ has lower asymptotic density $\alpha$ and upper asymptotic density $\beta$, then $\liminf \log p_A(n)/c_0 \sqrt{n} \geq \sqrt{\alpha}$ and $\limsup \log p_A(n)/c_0 \sqrt{n} \leq \sqrt{\beta}$. In particular, if $A$ has asymptotic density $\alpha > 0$, then $\log p_A(n) \sim c_0 \sqrt{\alpha n}$. Conversely, if $\alpha > 0$ and $\log p_A(n) \sim c_0 \sqrt{\alpha n}$, then the set $A$ has asymptotic density $\alpha$.

1. The growth of $p_A(n)$

Let $A$ be a nonempty set of positive integers. The counting function $A(x)$ of the set $A$ counts the number of positive elements of $A$ that do not exceed $x$. Then $0 \leq A(x) \leq x$, and so $0 \leq A(x)/x \leq 1$ for all $x$. The lower asymptotic density of $A$ is

$$d_L(A) = \liminf_{x \to \infty} \frac{A(x)}{x}.$$ 

The upper asymptotic density of $A$ is

$$d_U(A) = \limsup_{x \to \infty} \frac{A(x)}{x}.$$ 

We have $0 \leq d_L(A) \leq d_U(A) \leq 1$ for every set $A$. If $d_L(A) = d_U(A)$, then the limit

$$d(A) = \lim_{x \to \infty} \frac{A(x)}{x}$$

exists, and is called the asymptotic density of the set $A$.

A partition of $n$ with parts in $A$ is a representation of $n$ as a sum of not necessarily distinct elements of $A$, where the number of summands is unrestricted. The summands are called the parts of the partition. The partition function $p_A(n)$ counts the number of partitions of $n$ into parts belonging to

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the set $A$. Two partitions that differ only in the order of their parts are counted as the same partition. We define $p_A(0) = 1$ and $p_A(-n) = 0$ for $n \geq 1$.

The partition function for the set of all positive integers is denoted $p(n)$. Clearly, $0 \leq p_A(n) \leq p(n)$ for every integer $n$ and every set $A$. A classical result of Hardy and Ramanujan [4] and Uspensky [11] states that

$$\log p(n) \sim c_0\sqrt{n}, \quad \text{where} \quad c_0 = \pi \sqrt{\frac{2}{3}} = 2\sqrt{\frac{\pi^2}{6}}.$$

Erdős [2] has given an elementary proof of this result.

Let $\gcd(A)$ denote the greatest common divisor of the elements of $A$. If $d = \gcd(A) > 1$, consider the set $A' = \{a/d : a \in A\}$. Then $A'$ is a nonempty set of positive integers such that $\gcd(A') = 1$, and

$$p_A(n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod{d}, \\ p_{A'}(n/d) & \text{if } n \equiv 0 \pmod{d}. \end{cases}$$

Thus, it suffices to consider only partition functions for sets $A$ such that $\gcd(A) = 1$.

In this paper we investigate the relationship between the upper and lower asymptotic densities of a set $A$ and the asymptotic behavior of $\log p_A(n)$. In particular, we give a complete and elementary proof of the theorem that, for $\alpha > 0$, the set $A$ has density $\alpha$ if and only if $\log p_A(n) \sim c_0\sqrt{\alpha n}$. This result was stated, with a sketch of a proof, in a beautiful paper of Erdős [2].

Many other results about the asymptotics of partition functions can be found in Andrews [1, Chapter 6] and Odlyzko [8].

2. Some lemmas about partition functions

**Lemma 1.** Let $A$ be a set of positive integers. If $p_A(n_0) \geq 1$, then $p_A(n + n_0) \geq p_A(n)$ for every nonnegative integer $n$.

**Proof.** The inequality is true for $n = 0$, since $p_A(n_0) \geq 1 = p_A(0)$. We fix one partition $n_0 = a'_1 + \cdots + a'_r$. Let $n \geq 1$. To every partition $n = a_1 + \cdots + a_k$ we associate the partition

$$n + n_0 = a_1 + \cdots + a_k + a'_1 + \cdots + a'_r.$$

This is a one-to-one map from partitions of $n$ to partitions of $n + n_0$, and so $p_A(n) \leq p_A(n + n_0)$. 

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