Secondary Deformations Due to Axial Shear of the Annular Region Between Two Eccentrically Placed Cylinders

Dedicated to Prof. Roger L. Fosdick on his sixtieth birthday

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Abstract. Fosdick and Kao [1] extended a conjecture of Ericksen’s [2] for non-linear fluids, to non-linear elastic solids, and showed that unless the material moduli of an isotropic elastic material satisfied certain special relations, axial shearing of cylinders would be necessarily accompanied by secondary deformations if the cross-section were not a circle or the annular region between two concentric circles. Further, they used the driving force as the small parameter for a perturbation analysis and showed that the secondary deformation will occur at fourth order, much in common with what is known for non-linear fluids. Here, we show that if on the other hand the driving force is not small (of $O(1)$), but the departure of the cylinder from circular symmetry is small, then secondary deformations appear at first order, the parameter for perturbation being the divergence from circular symmetry.

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1. Introduction

A common underlying feature of the response of many non-linear materials is the manifestation of normal stress differences, (in simple shear in solids, in shearing in fluids or granular materials), the associated phenomena being referred to as the Poynting effect in solid theories, Weissenberg effect in fluid theories and Dilatancy in theories for granular materials. The implications of the presence of normal stress differences are manifold: they are the cause of phenomena such as ‘die-swell’ and ‘rod-climbing’ that are displayed by many non-linear fluids; corresponding behavior being possible in non-linear solids and granular materials. Here we shall discuss yet another demonstration of normal stress differences: the occurrence of secondary deformations in solids (similar to that of secondary flows in fluids).

Within the context of non-linear fluids, Ericksen [2] conjectured that purely rectilinear flows would be possible only in pipes of circular cross-section or cross-sections made of straight lines and circles, secondary flows being necessarily
present in pipes of arbitrary cross-sections. A more precise version of the conjecture was rigorously proved by Fosdick and Serrin [3]; they showed that unless the material functions characterizing the fluid satisfied certain special relationships, the cross-section ought to be a circle or the annular region between two concentric circles. They also required certain technical assumptions on regularity concerning the material properties as a function of the shearing.

Langlois and Rivlin [4] and Green and Rivlin [5] had earlier studied the secondary flow patterns for a special class of fluids, and later Truesdell and Noll [6] provided a systematic perturbation approach to study such secondary flows based on the assumption that the ‘driving force’ is small and that a general simple fluid can be approximated in a simple manner in retarded motions. As pointed out recently by Dunn and Rajagopal [7] the retardation procedure does not provide a hierarchy of models. That this formal approach has a serious shortcoming with regard to solving general flows involving complex fluids was recognized by Truesdell and Noll [6] who observe that ‘It is plausible that the flow will be slow when “a” is small. However, there is no reason to believe that the flow for a small specific driving force can be obtained from a larger specific driving force by a mere retardation. Therefore the asymptotic approximations for the flow discussed in Section 40 do not apply directly.’ However, they proceed to use the retarded motion approximations for the problem under consideration to get some insight into the problem, as they recognize that a proof of convergence for the problem would be most daunting, if at all possible. A similar procedure that is formal, with no proofs of convergence, has been used to study longitudinal and torsional oscillations in a simple fluid (cf. Rajagopal, Kaloni and Tao [8]).

Interestingly, secondary flows accompanying the turbulent motion of fluids in pipes of non-circular cross-sections were observed as early as 1926 by Nikuradse [9 and 10]. Experiments show that the magnitude of the turbulent secondary velocity is about three percent of the mean axial velocity. Since Nikuradse’s early studies there have been numerous experimental measurements of secondary flow in pipes of non-circular cross-section (see Huang and Rajagopal [11]). Such secondary flows in turbulence motivated Rivlin [12] to conjecture a similarity in the modelling of turbulent flows of Newtonian fluids and laminar flows of non-Newtonian fluids, which in turn has led to an interesting class of turbulence models (see Speziale [13], Huang and Rajagopal [11]).

Fosdick and Kao [1] were the first to explore the counterpart to Ericksen’s conjecture in fluids, within the context of non-linear elastic solids. They showed

1 ‘a’ denotes the pressure gradient that is the driving force in the problem of flow down a pipe of elliptic cross-section studied by Truesdell and Noll [6].

2 The asymptotic approximations in Section 40 of Truesdell and Noll [6] refer to the representations that have the forms of grade n fluids.

3 Fosdick and Kao [1] did not consider the exact counterpart in that they restricted their analysis to elastostatics. However, as the underlying physical mechanism for the manifestation of the secondary effects is due to material non-linearities, such a neglect of the effect of inertia does not alter the conclusions to be reached.