ANISOTROPIC MATERIAL WITH ARBITRARILY ORIENTED CRACKS
AND ELLIPTICAL HOLES: EFFECTIVE ELASTIC MODULI

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Abstract. Effective moduli of a 2-D anisotropic solid with elliptical holes of an
arbitrary (non-random) orientational distribution are given in closed form. The results are
derived in the non-interacting approximation - the basic building block for various
approximate schemes. Proper tensorial parameters of defect density are identified.

1. Introduction. Strain per reference area $A$ containing a hole is a sum:
\[ \varepsilon = S^0 : \sigma + \Delta \varepsilon \] (or \( \varepsilon_{ij} = S_{ijkl}^0 \sigma_{kl} + \Delta \varepsilon_{ij} \) in indicial notation), where $S^0$ is the compliance
tensor of the matrix. Hole's contribution $\Delta \varepsilon$ is given by the integral
\[ \Delta \varepsilon = -\frac{1}{2A} \int_{\Gamma} \left( u_i n_j + u_j n_i \right) d\Gamma \] (1.1)
where $n$ is the unit normal to hole boundary $\Gamma$ (inwards the hole) and $u$ denotes
displacements of points of $\Gamma$ and $u n, u n$ are dyadic products of two vectors.

Due to linear elasticity, $\Delta \varepsilon$ is a linear function of $\sigma$ and hence can be written as
\[ \Delta \varepsilon = H : \sigma \] (1.2)
where fourth rank tensor $H$ is the cavity compliance tensor (possessing the usual
symmetries $H_{ijkl} = H_{jkil} = H_{ijlk}$). $H$-tensors were found for a number of 2-D and 3-D
cavity shapes by Tsukrov and Kachanov (1993) and Kachanov et al. (1994), in the case
of the isotropic matrix. Maugé and Kachanov (1994) derived results for an anisotropic
matrix with cracks of arbitrary orientations. Here, we bridge these two lines of analysis.

We formulate the problem in terms of the elastic potential (rather than compliances):
its structure implies the proper parameters of defect density and establishes the overall
anisotropy. The potential in stresses $f(\sigma) = (1/2) \sigma : \varepsilon(\sigma)$ of a solid with a hole is
\[
f(\sigma) = (1/2)\sigma : S^0 : \sigma + (1/2)\sigma : H : \sigma = f_0 + \Delta f \tag{1.3}
\]

For the orthotropic solid, \(f_0(\sigma_0) = \left(1/2E_1^0\right)\sigma_{11}^2 + \left(1/2E_2^0\right)\sigma_{22}^2 - \left(\nu_0^0 / E_1^0\right)\sigma_{11}\sigma_{22} + \left(1/2G_1^0\right)\sigma_{12}^2\) where \(E_1^0\), \(G_1^0\), and \(\nu_0^0\) are Young's moduli, shear moduli and Poisson's ratios of the matrix in the case of plane stress. In plane strain, \(E_1 \rightarrow \frac{E_1}{(1 - \nu_1^3\nu_3)}, E_2 \rightarrow \frac{E_2}{(1 - \nu_2^3\nu_3)}, \nu_1 \rightarrow \frac{\nu_1 + \nu_3\nu_3}{(1 - \nu_3^3)}, \nu_2 \rightarrow \frac{\nu_2 + \nu_3\nu_3}{(1 - \nu_3^3)}\).

In the case of a crack, (1.1) reduces to \(\Delta \varepsilon = \left(l^2/A\right)(bn + nb)\), where vector \(b = \left(u_+ - u_+^{-}\right)/l\) is the average over the crack displacement discontinuity normalized to \(l\).

Due to linear elasticity, \(b\) is a linear function of traction \(n \cdot \sigma\) induced by remotely applied \(\sigma\) at the crack site in a continuous material:

\[
b = n \cdot \sigma \cdot B \quad \text{(or } b_i = \sigma_{ik}n_k B_{ji}) \tag{1.4}\]

where symmetric second rank tensor \(B\), introduced by Kachanov (1992), is the crack compliance tensor. \(B\) - tensors for 3-D elliptical cracks, cracks constrained against the normal opening, fluid filled cracks were derived by Kachanov (1992, 1993), for cracks arbitrarily oriented in 2-D anisotropic matrix - by Mauge and Kachanov (1992, 1994).

Thus, in the change in the potential due to 2-D rectilinear crack is

\[
\Delta f = (1/2)\sigma : \Delta \varepsilon = (1/2)\sigma : l^2 n B n : \sigma \tag{1.5}
\]

hence identifying \(H\)-tensor of a crack as (appropriately symmetrized)

\[
H = (2l^2/A) n B n \quad \text{i.e. } H_{ijkl} = \left(l^2/2\right)\left(n_i B_{kj} n_k + n_k B_{ij} n_i + n_l B_{ij} n_l + n_l B_{kl} n_k\right) \tag{1.6}
\]

We now consider a matrix with many holes in the non-interaction approximation: each hole is placed into \(\sigma\)-field and does not experience any influence of neighbors. This approximation is of the fundamental importance: besides being rigorous at small defect densities, it constitutes the basic building block for various approximate.

In the non-interaction approximation, the potential change due to holes is

\[
\Delta f = (1/2)\sigma : \sum H^{(k)} : \sigma \tag{1.7}
\]

Tensor \(H = \sum H^{(k)}\) (summation may be replaced by integration over orientations, if computationally convenient) takes the individual cavity contributions with proper "relative weights" and, thus, constitutes the proper parameter of defect density. Below, we specialize it for elliptical holes in the orthotropic matrix. The effective elastic compliances \(S_{ijkl}\) are obtained from \(e_{ij} = \partial(f_0 + \Delta f) / \partial \sigma_{ij} = S_{ijkl} \sigma_{kl}\).

In the case of cracks, (1.7) takes the form